How delay equations arise in Engineering?

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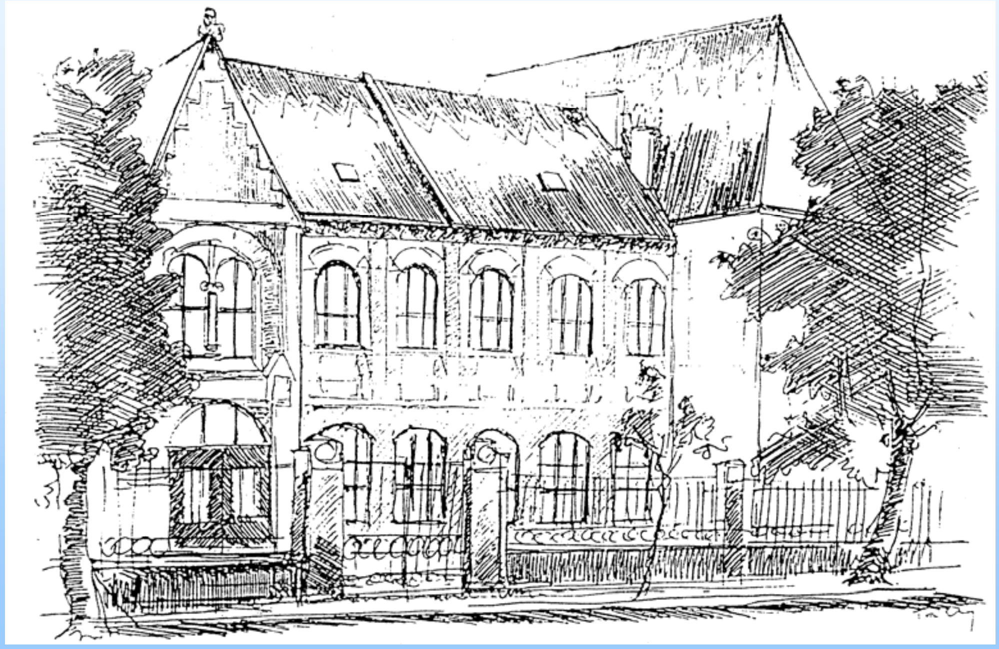


Contents

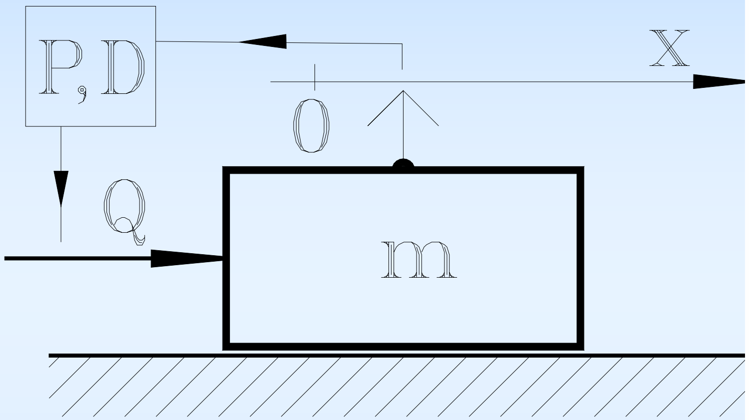
Answer: Delay equations arise in Engineering…

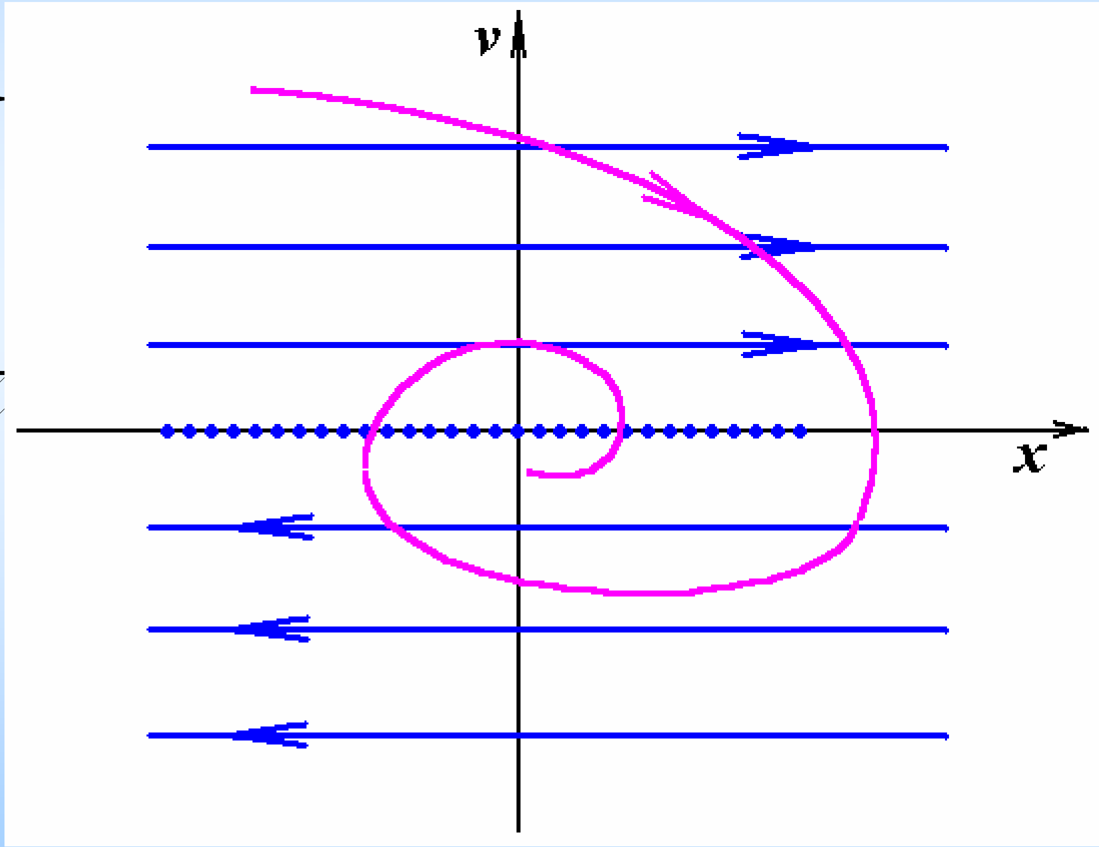
… by the information system (of control), and
by the contact of bodies.

- Linear stability & subcritical Hopf bifurcations
- Robotic position and force control
- **Balancing – human and robotic**
- Contact problems
- Shimmying wheels (of trucks and motorcycles)
- Machine tool vibrations

Position control

1 DoF models $\Rightarrow$ x





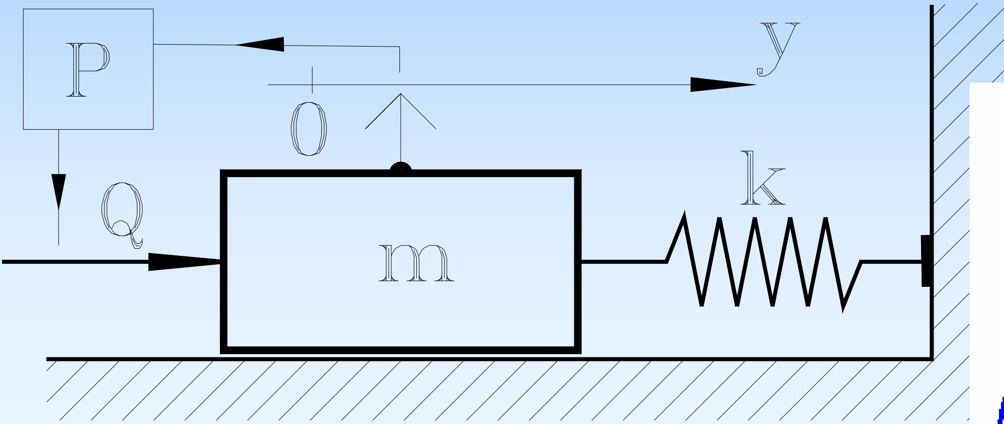
Blue trajectories:

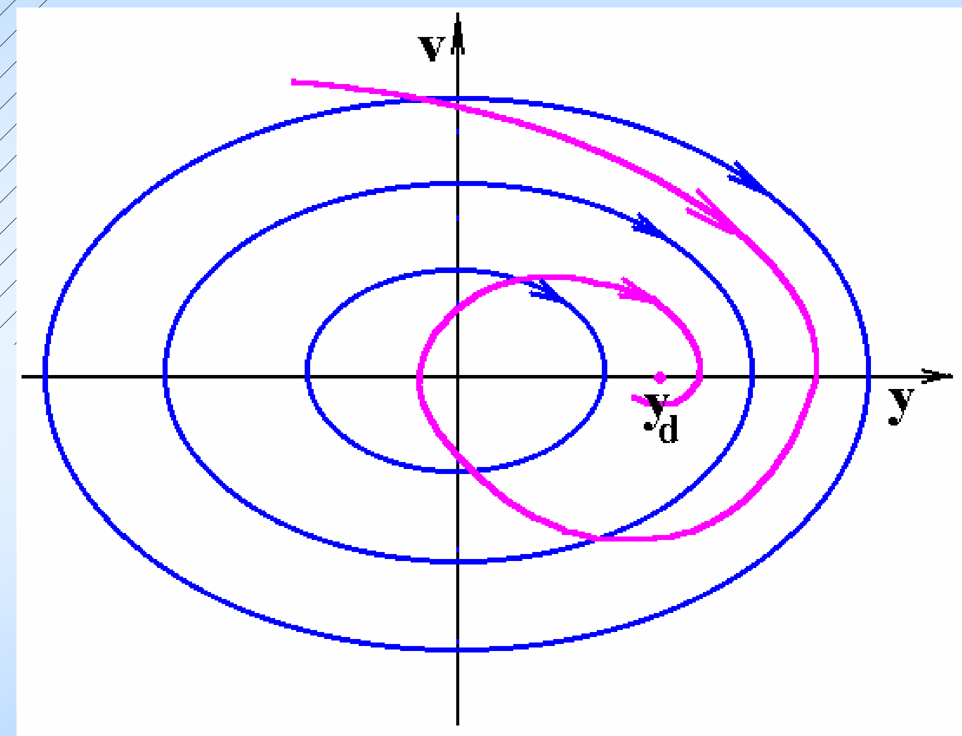
$Q = 0$

Pink trajectories:

$Q = -Px - D\dot{x}$

Force control





Desired contact force:

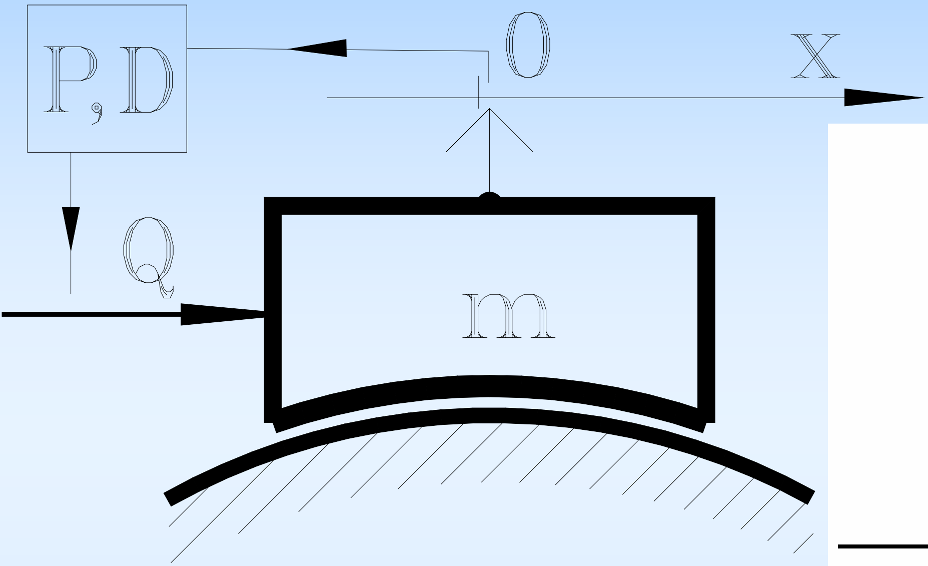
$$F_d = ky_d\ ;$$

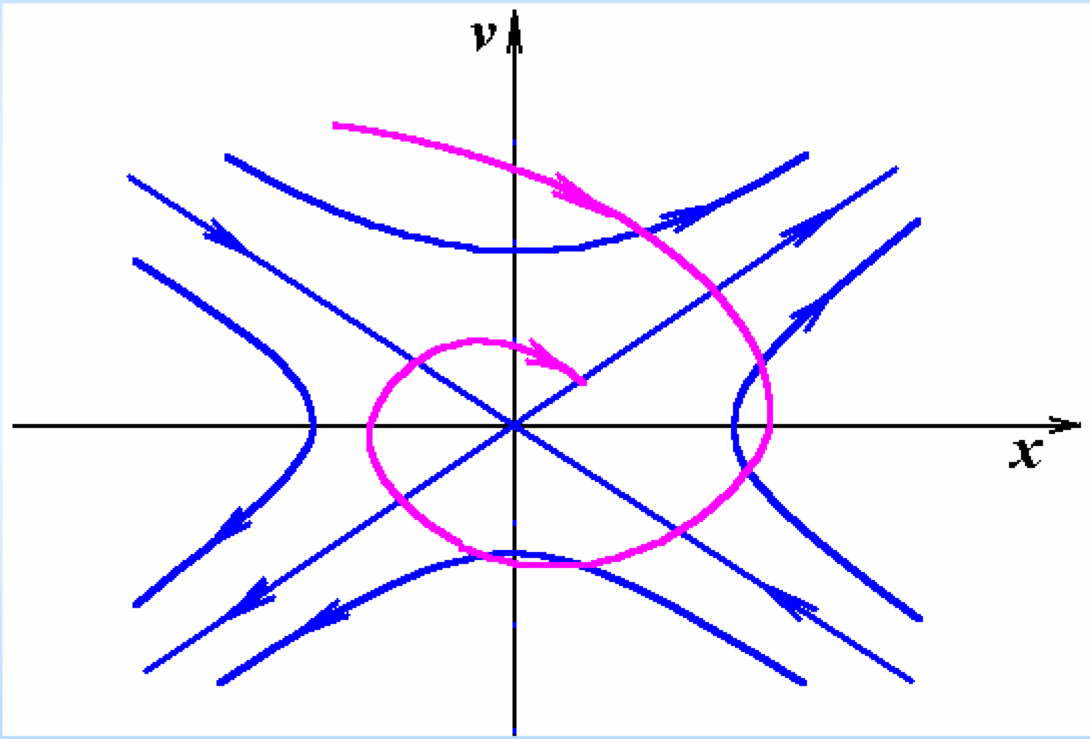
Sensed force:

$$F_s = ky$$

Control force: $Q = -P(F_d - F_s) - D\dot{F}_s + F_{s\ \text{or}\ d}$

Stabilization (balancing)





Control force:

$$Q = -Px - D\dot{x}$$

Special case of force control: with $k < 0$

Modeling digital control

Special cases of force control:

- position control with zero stiffness ($k = 0$)
- stabilization with negative stiffness ($k < 0$)

Digital effects:

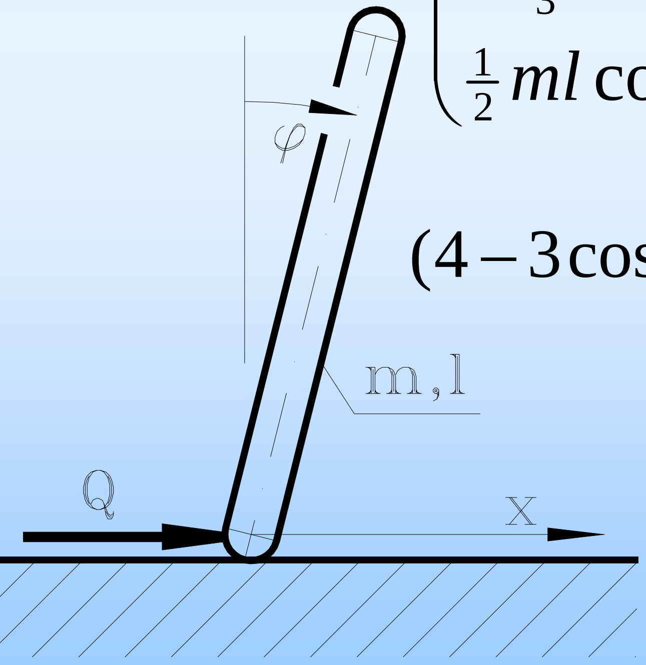
- quantization in time: *sampling* – **linear**
- quantization in space: *round-off* errors
  at ADA converters
  – **non-linear**

Balancing inverted pendulum

Higdon, Cannon (1962) …10-20 papers / year

$n = 2$ DoF $\Rightarrow$ φ, x ; x – cyclic coordinate

$$\begin{pmatrix} \frac{1}{3}ml^2 & \frac{1}{2}ml\cos\varphi \\ \frac{1}{2}ml\cos\varphi & m \end{pmatrix} \begin{pmatrix} \ddot{\varphi} \\ \ddot{x} \end{pmatrix} - \begin{pmatrix} \frac{1}{2}mgl\sin\varphi \\ \frac{1}{2}ml^2\dot{\varphi}^2\sin\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ Q \end{pmatrix}$$



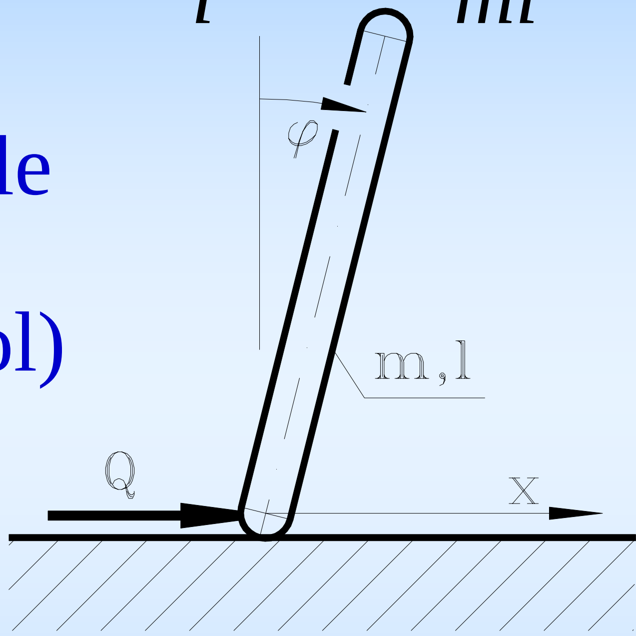
$$(4 - 3\cos^2\varphi)\ddot{\varphi} + \frac{3}{2}\dot{\varphi}^2\sin(2\varphi) - 6\frac{g}{l}\sin\varphi = -6\frac{Q}{ml}\cos\varphi$$

linearization at $\varphi = 0$

$$\boxed{\ddot{\varphi} - 6\frac{g}{l}\varphi = -\frac{6}{ml}Q}$$

Balancing

$$\ddot{\varphi} - 6\frac{g}{l}\varphi = -\frac{6}{ml}Q$$



1) $Q = 0$ - no control

$$\ddot{\varphi} - 6\frac{g}{l}\varphi = 0 \;\Rightarrow\; \varphi = 0 \text{ is unstable}$$

2) $Q(t) = P\varphi(t) + D\dot{\varphi}(t)$ (PD control)

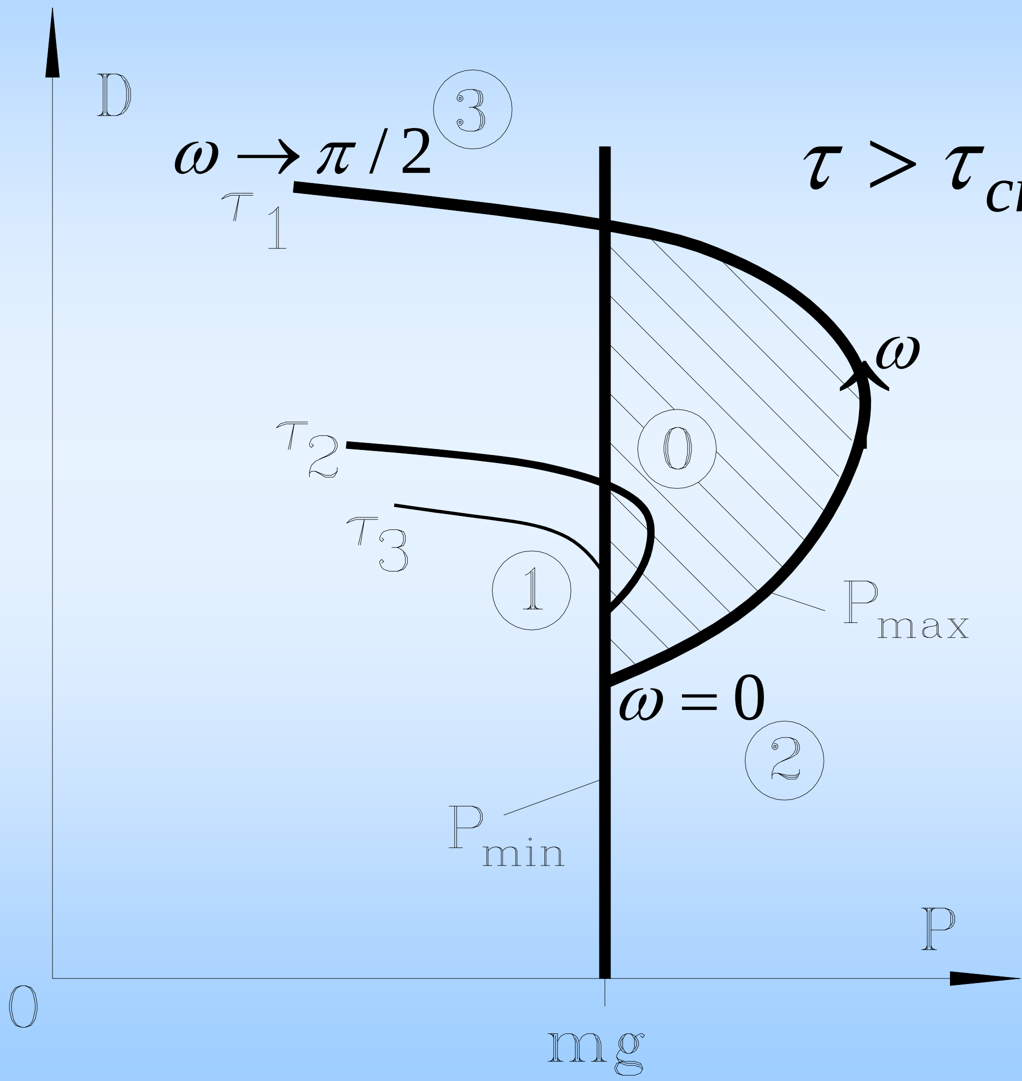
$$\ddot{\varphi} + \frac{6}{ml}D\dot{\varphi} + \frac{6}{ml}(P - mg)\varphi = 0$$

$\varphi = 0$ is asympt. stable $\Leftrightarrow D > 0,\; P > mg$

3) $Q(t) = P\varphi(t-\tau) + D\dot{\varphi}(t-\tau)$ (with reflex delay τ)

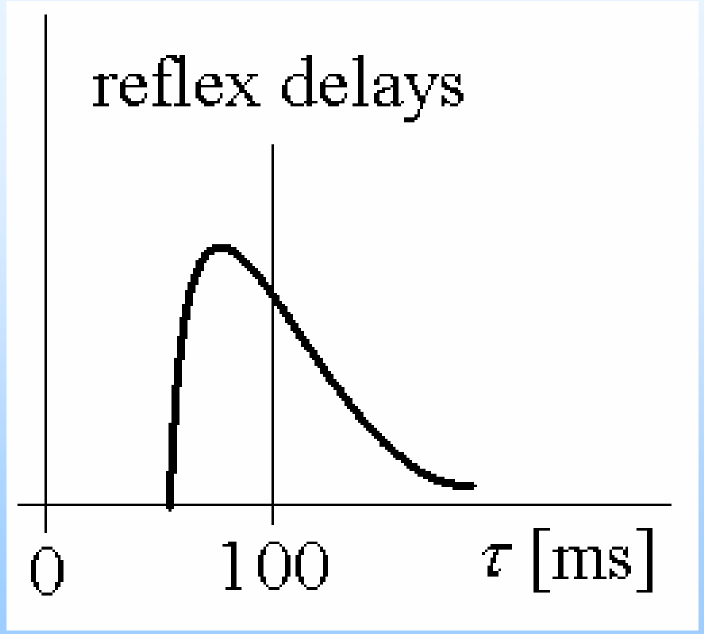
$$\ddot{\varphi}(t) + \frac{6}{ml}D\dot{\varphi}(t-\tau) + \frac{6}{ml}P\varphi(t-\tau) - \frac{6g}{l}\varphi(t) = 0$$

Stability chart & critical reflex delay

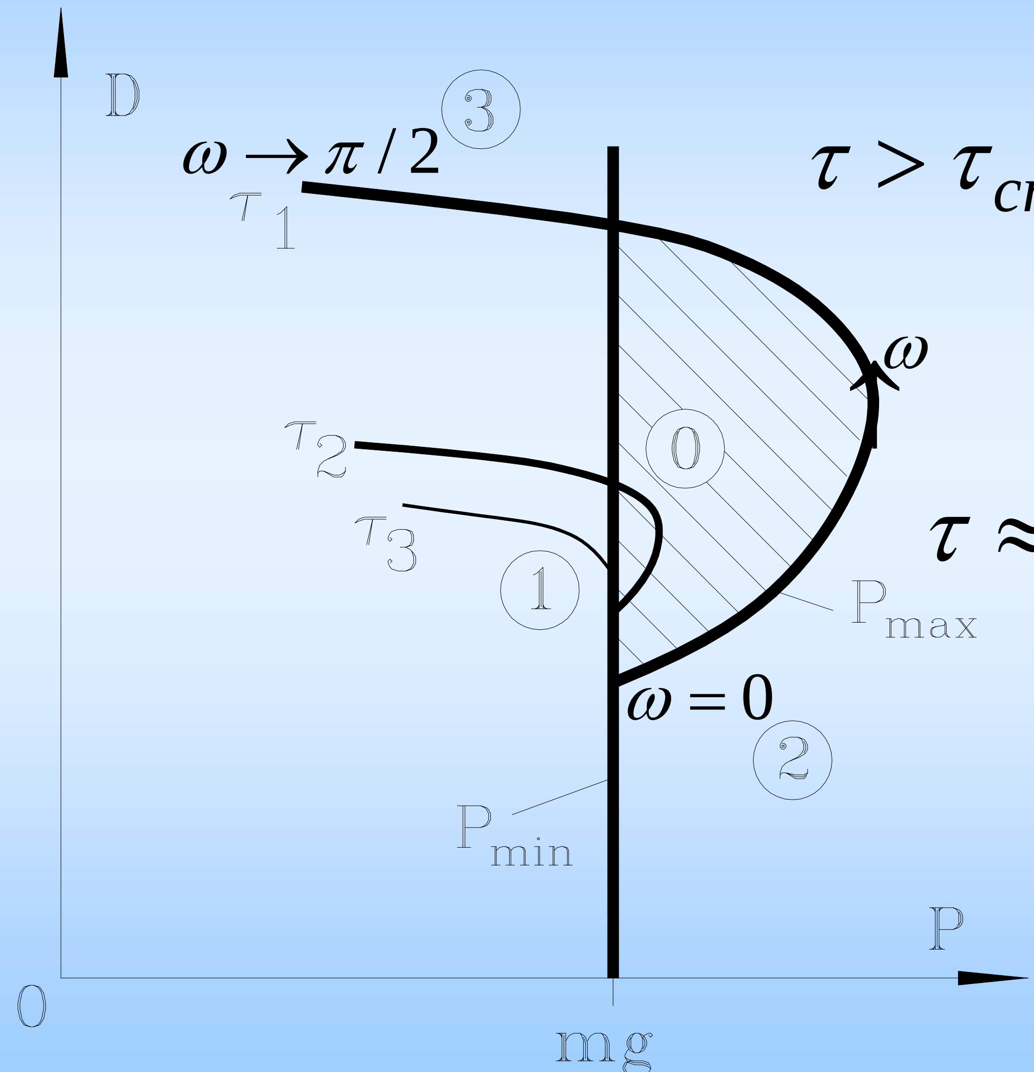


$$\tau > \tau_{cr} = \sqrt{\frac{l}{3g}} \Rightarrow \text{instability}$$

$$l = 0.3\,[\text{m}]$$



Stability chart & critical reflex delay

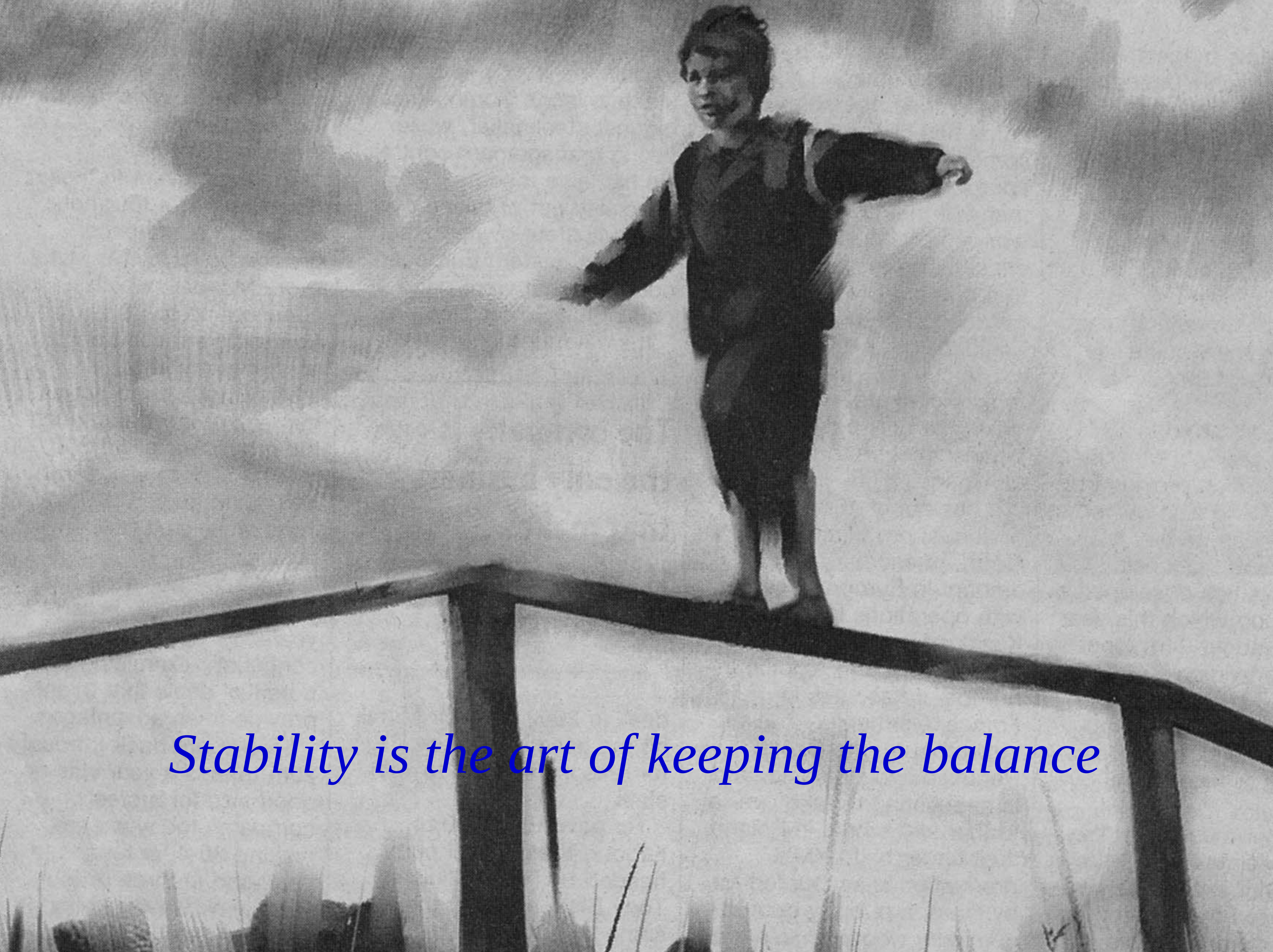


$$\tau > \tau_{cr} = \sqrt{\frac{l}{3g}}$$ ⇒ instability

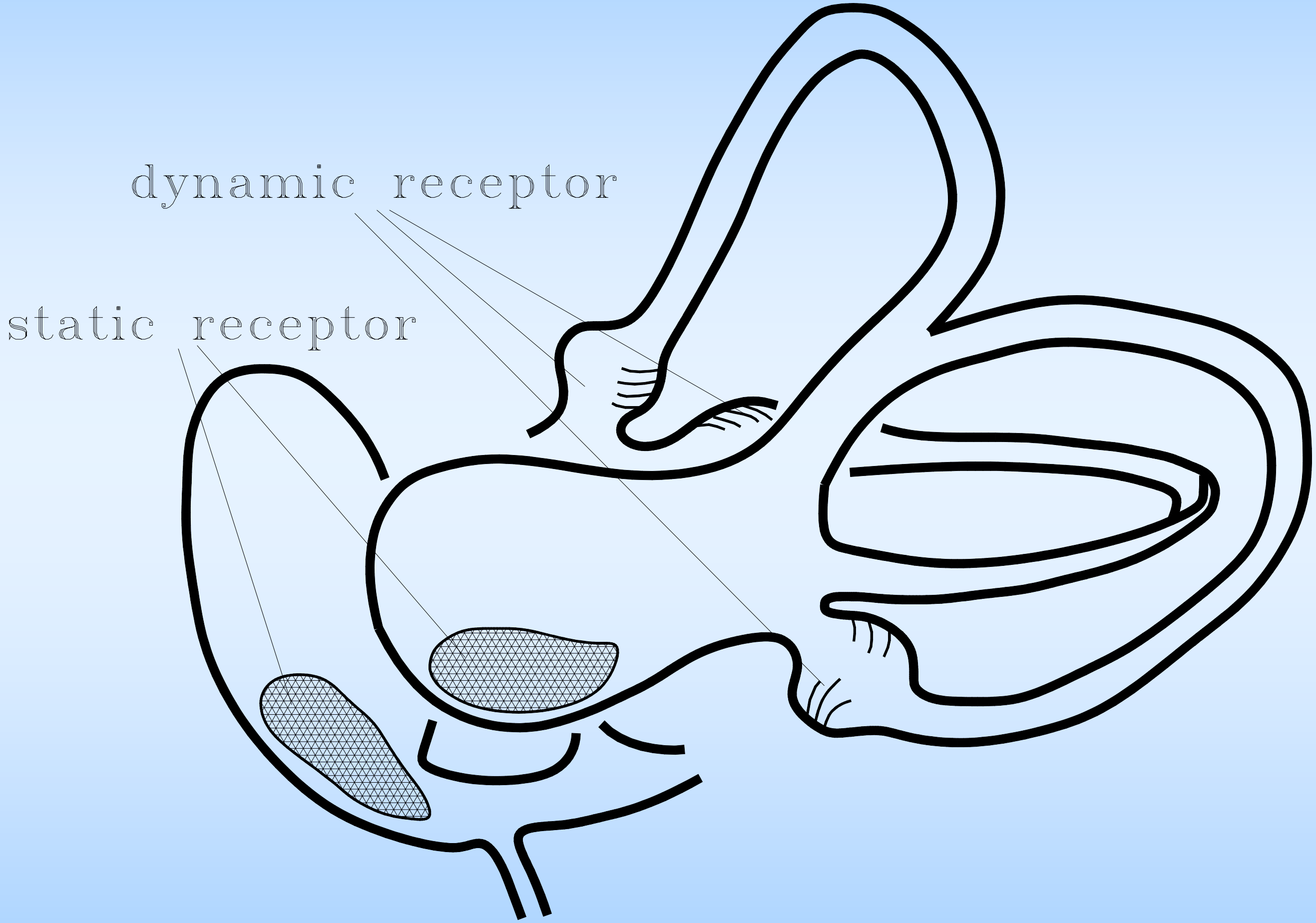
$$l = 0.3\,[\mathrm{m}]$$

$$\tau \approx \sqrt{0.3/(3\cdot 10)} = 0.1\,[\mathrm{s}]$$

$$0 < \frac{\omega}{2\pi\tau} = f < \frac{1}{4\tau} \approx 2.5\,[\mathrm{Hz}]$$


Stability is the art of keeping the balance

Labyrinth – human balancing organ



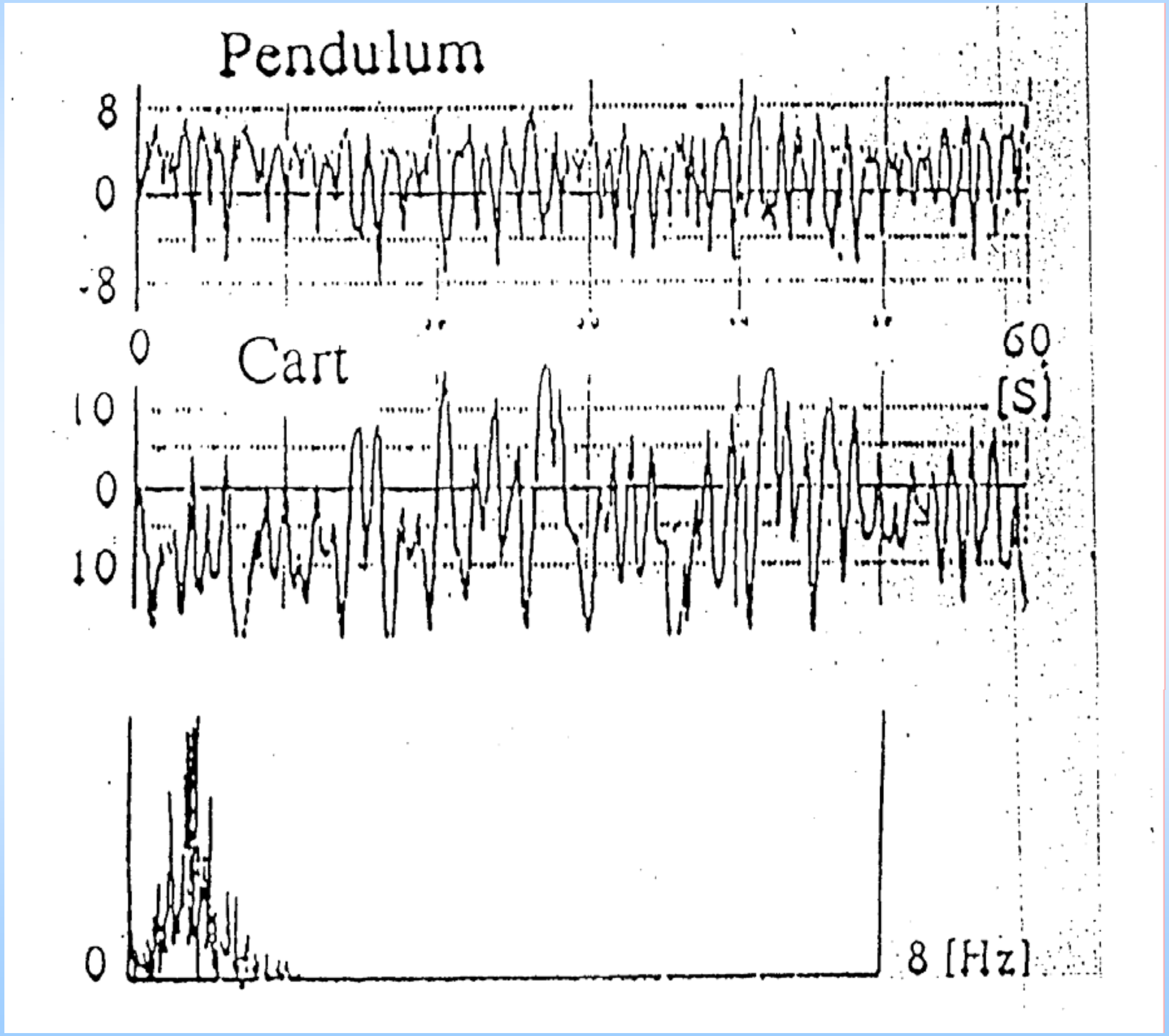
Both angle and angular velocity signals are needed

irvingmm.avi

Experimental observations

Kawazoe (1992)
- untrained
- manual
- control

Betzke (1994)
- target
- shooting
- 0.3 – 0.7 [Hz]

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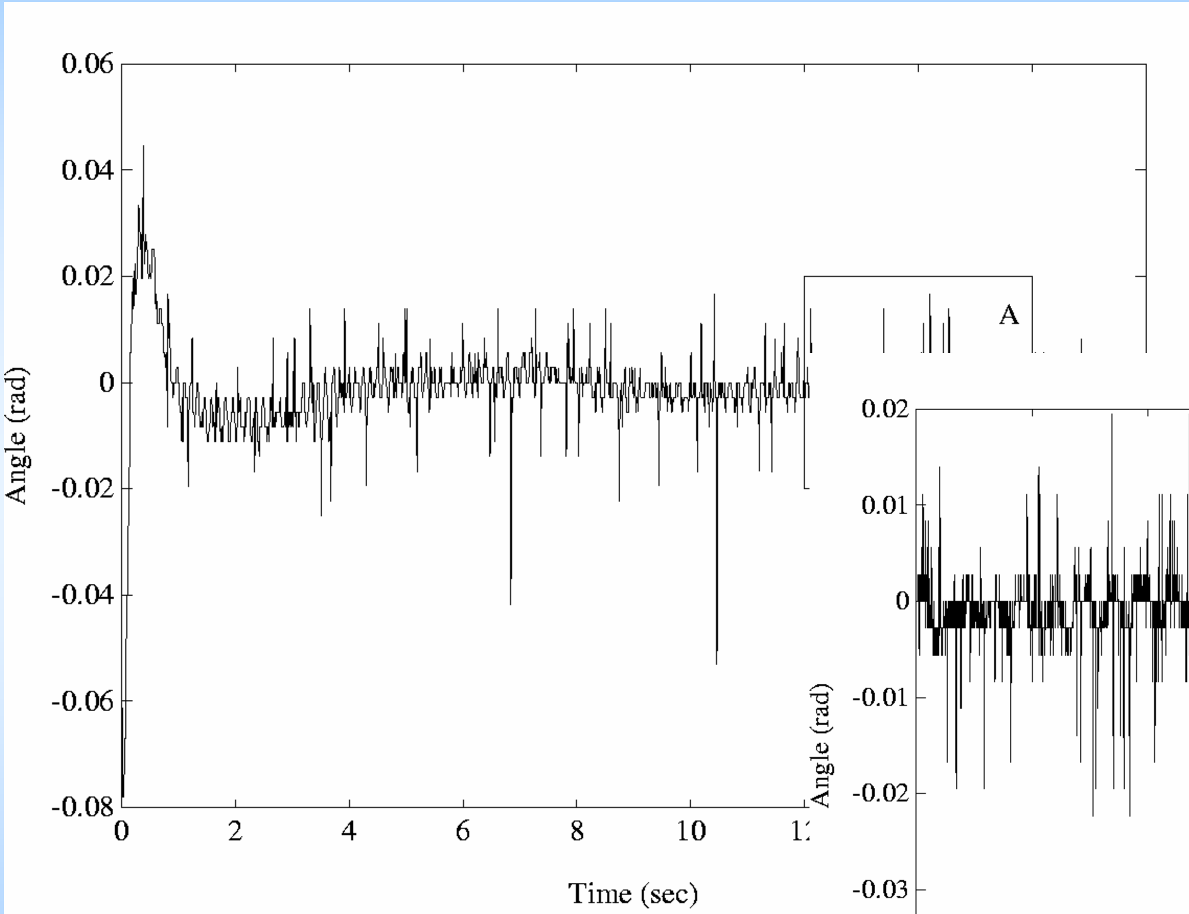
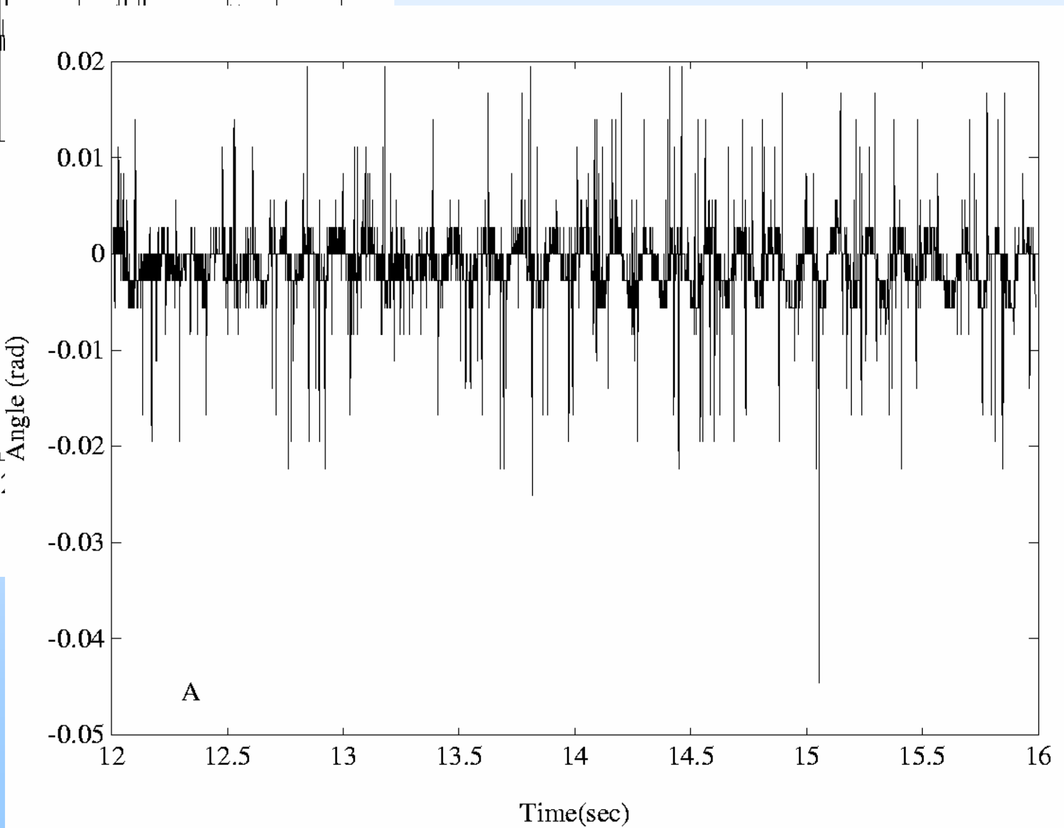
Nincs kép.

roller.mpg

Random oscillations of robotic balancing

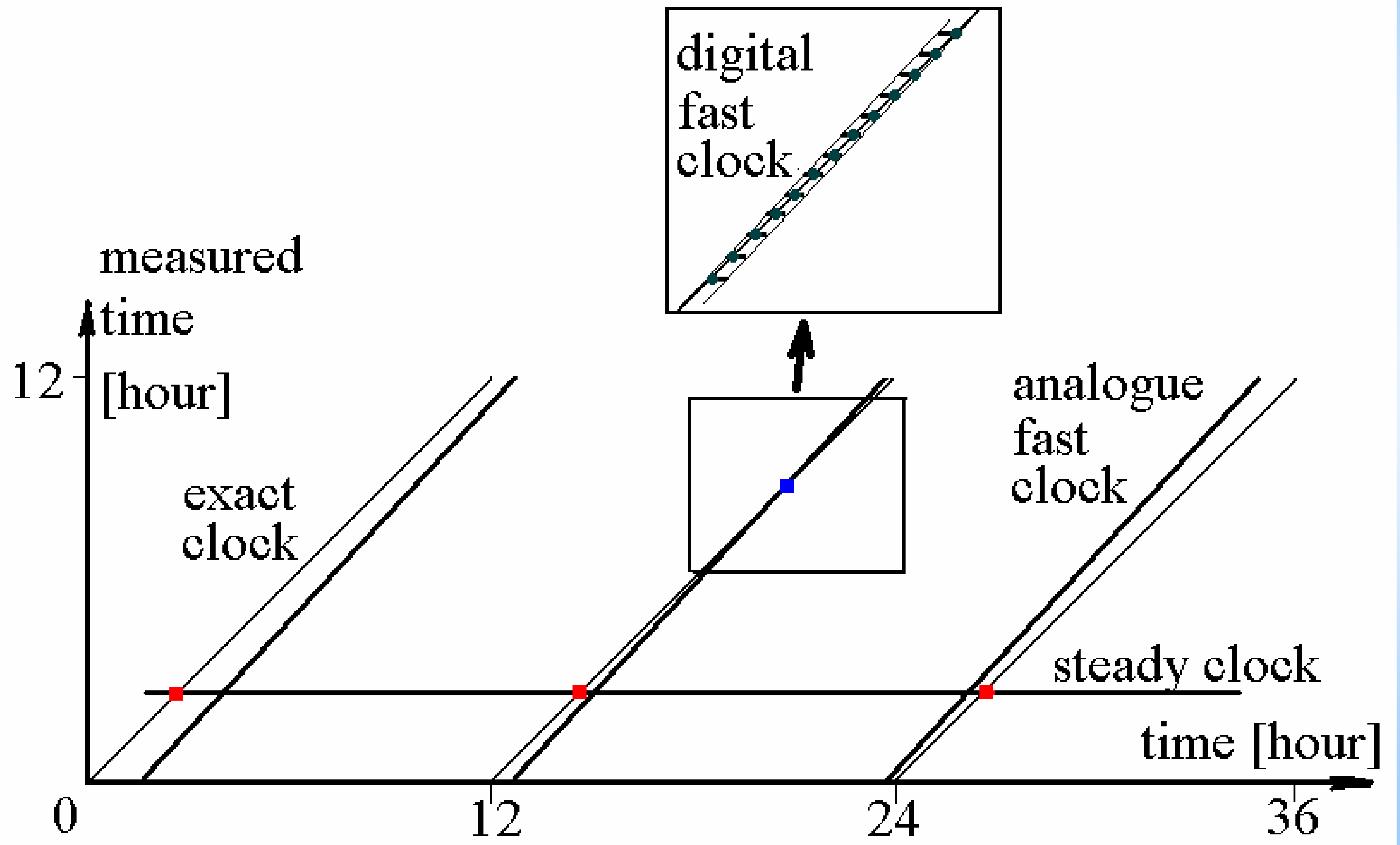


sampling time
and

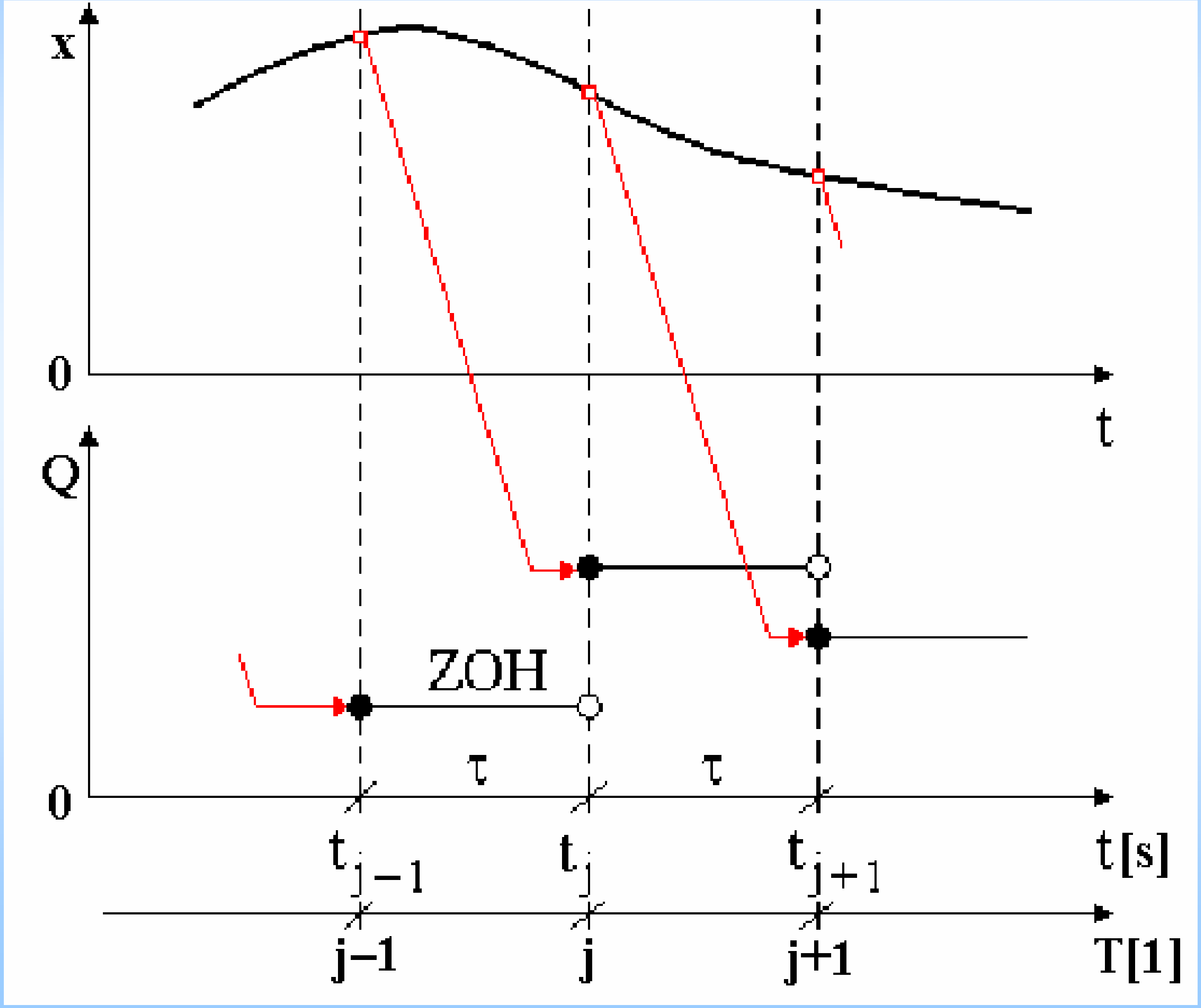


quantization
(round-off)

Alice's Adventures in Wonderland



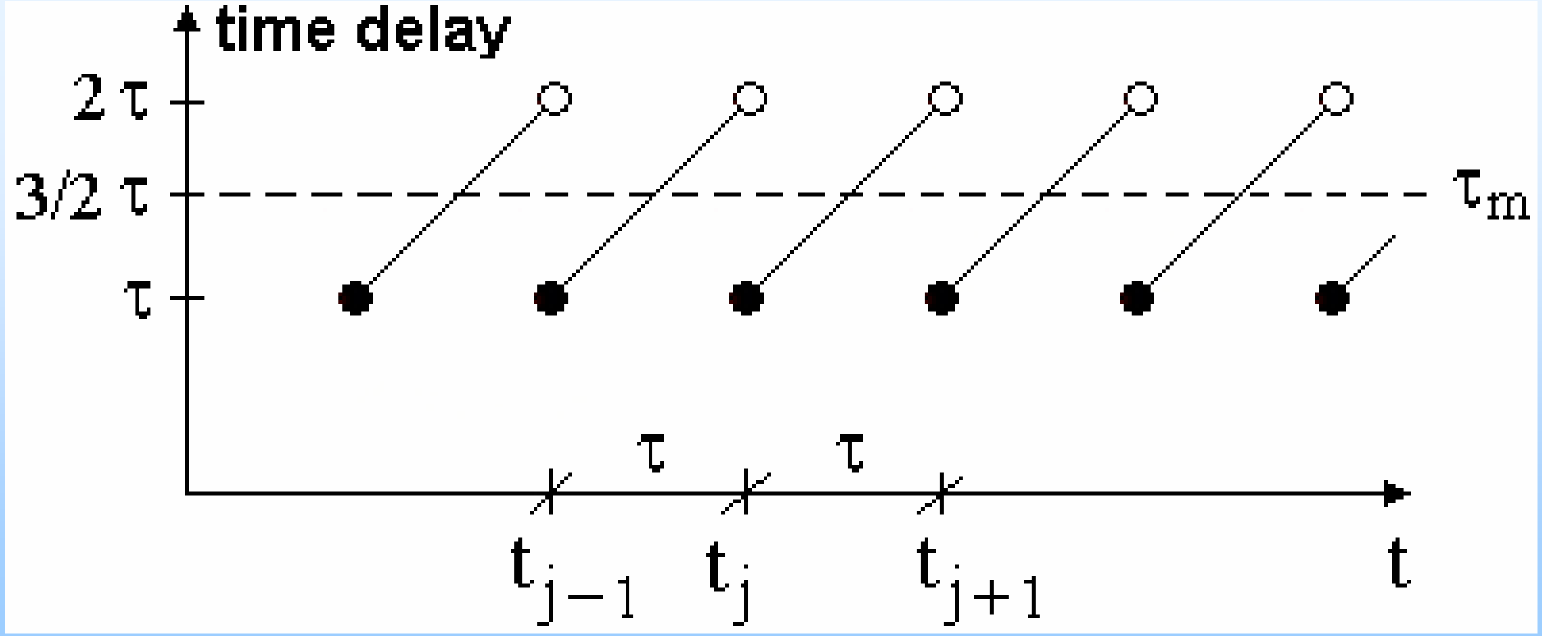
Sampling delay of digital control



Digitally controlled pendulum

$$\ddot{\varphi}(t) - \frac{6g}{l}\varphi(t) = u_j \,, \quad t \in [t_j, t_j + \tau)$$

$$u_j = -\frac{6}{ml}\left(D\dot{\varphi}(t_j - \tau) + P\varphi(t_j - \tau)\right) \,, \quad j = 1, 2, \ldots$$



Stability of digital control – sampling

$$\mathbf{x}^j = \begin{pmatrix} \varphi(t_j) \\ \dot{\varphi}(t_j) \\ u_j \end{pmatrix}$$

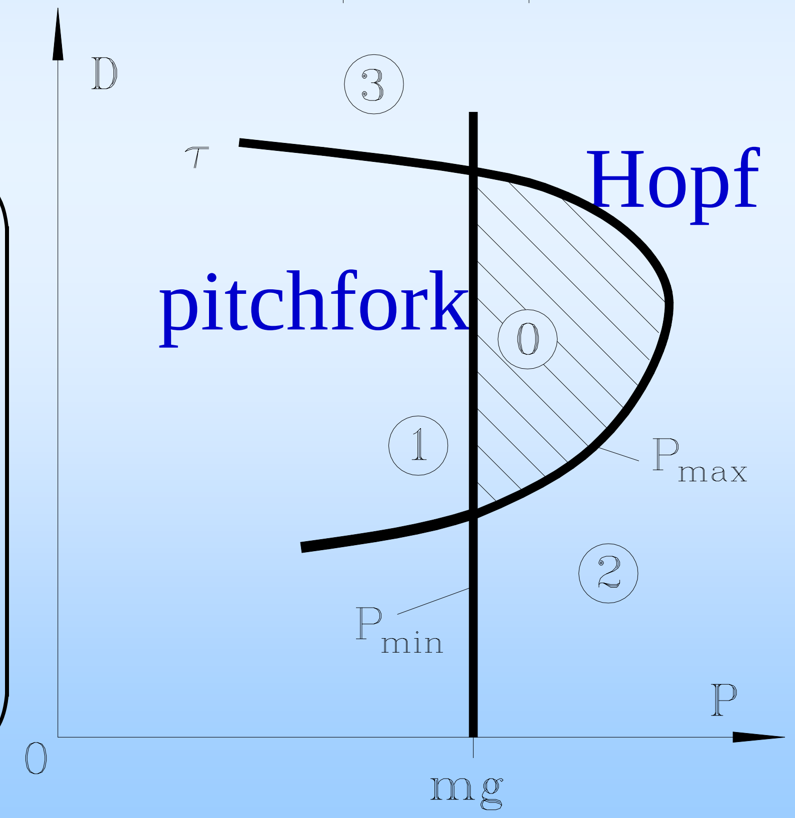
$$\mathbf{x}^{j+1} = \mathbf{A}\mathbf{x}^j$$

$$\det(\lambda \mathbf{I} - \mathbf{A}) = 0$$

$$|\lambda_{1,2,3}| < 1$$

$$\omega = \tau \sqrt{\frac{6g}{l}}$$

$$\mathbf{A} = \begin{pmatrix} \operatorname{ch}\omega & \dfrac{\operatorname{sh}\omega}{\omega} & \dfrac{\operatorname{ch}\omega - 1}{\omega^2} \\ \omega \operatorname{sh}\omega & \operatorname{ch}\omega & \dfrac{\operatorname{sh}\omega}{\omega} \\ -\dfrac{6\tau^2}{ml} P & -\dfrac{6\tau}{ml} D & 0 \end{pmatrix}$$



Stability of digital control – round-off

h – one digit converted to control force

$$u_j = -\frac{6}{ml} h \,\mathrm{int}\left(\frac{D\dot{\varphi}(t_j - \tau) + P\varphi(t_j - \tau)}{h} \right)$$

$$\mathbf{x}^{j+1} = \mathbf{B}\mathbf{x}^j + \mathbf{g}(\mathbf{x}^j)$$

$$\mathbf{g}(\mathbf{x}^j) = \begin{pmatrix} 0 \\ 0 \\ -\frac{6\tau^2}{ml} h \,\mathrm{int}\left(\frac{P}{h} x_1^j + \frac{D}{\tau h} x_2^j \right) \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \mathrm{ch}\,\omega & \frac{\mathrm{sh}\,\omega}{\omega} & \frac{\mathrm{ch}\,\omega - 1}{\omega^2} \\ \omega\,\mathrm{sh}\,\omega & \mathrm{ch}\,\omega & \frac{\mathrm{sh}\,\omega}{\omega} \\ 0 & 0 & 0 \end{pmatrix}$$

$\det(\lambda \mathbf{I} - \mathbf{B}) = 0 \Rightarrow$

$\lambda_1 = e^{\omega} > 1$, $\lambda_2 = e^{-\omega}$, $\lambda_3 = 0$

1D cartoon – the μ-chaos map

Drop 2 dimensions, rescale x with h $\Rightarrow$ a ~ e$^{\omega}$,
$b \sim P$

$$x_{j+1} = a x_j - b\,\mathrm{int}(x_j)$$

A pure mathematical approach ($p > 0$, $p < q$)

$$\dot{y}(t) = p y(t) - q\,\mathrm{int}\big(y\big(\mathrm{int}(t)\big)\big)$$

solution with $x_j = y(j)$ leads to μ-chaos map,

$a = e^p$, $b = q(e^p - 1)/p$ $\Rightarrow$ $a > 1$, $(0 <)\ a - b < 1$

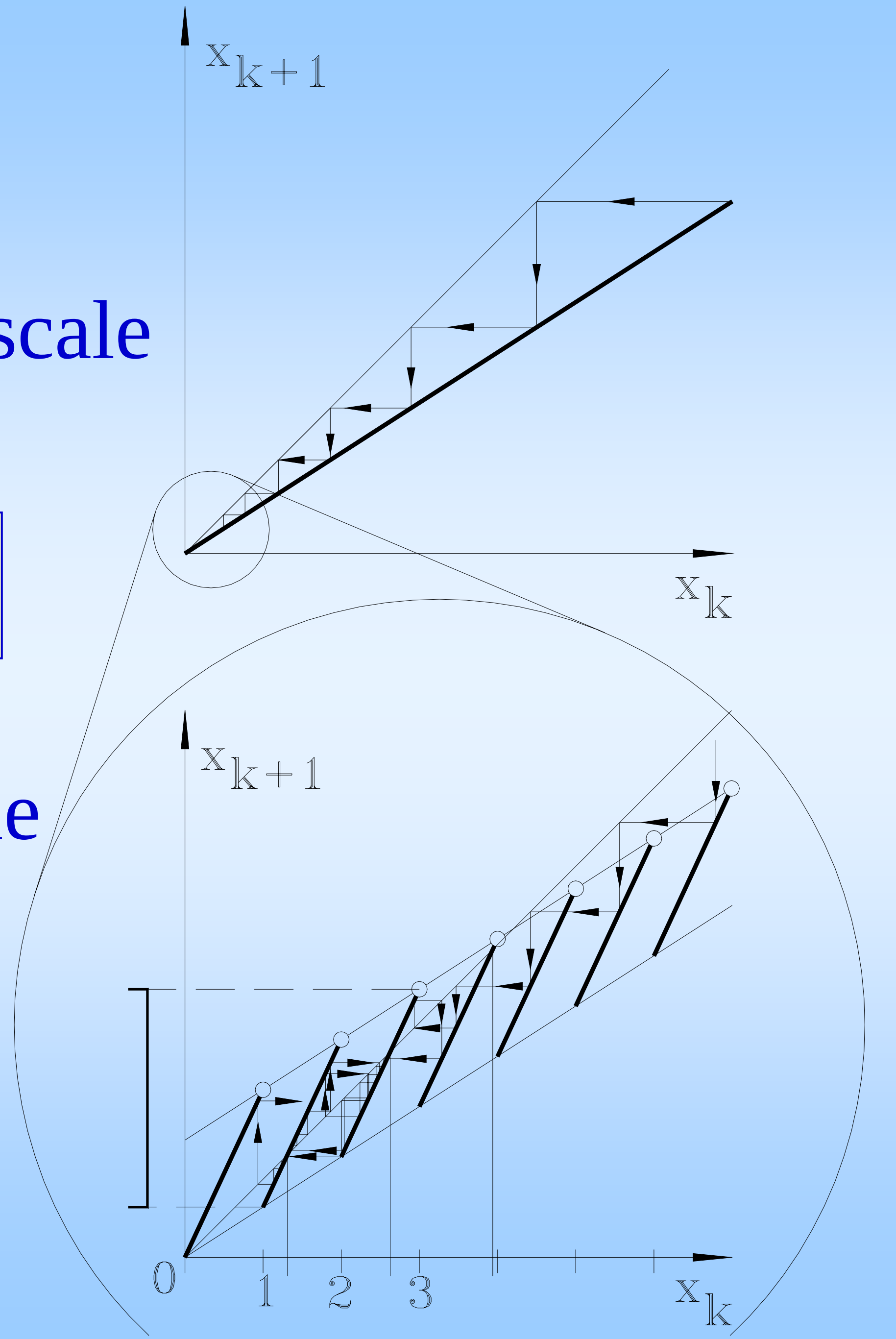
small scale: $x_{j+1} = a\, x_j$, large scale: $x_{j+1} = (a - b)\, x_j$

Micro-chaos map

large scale

$$x_{k+1} = a x_k - b \,\mathrm{int}(x_k)$$

small scale

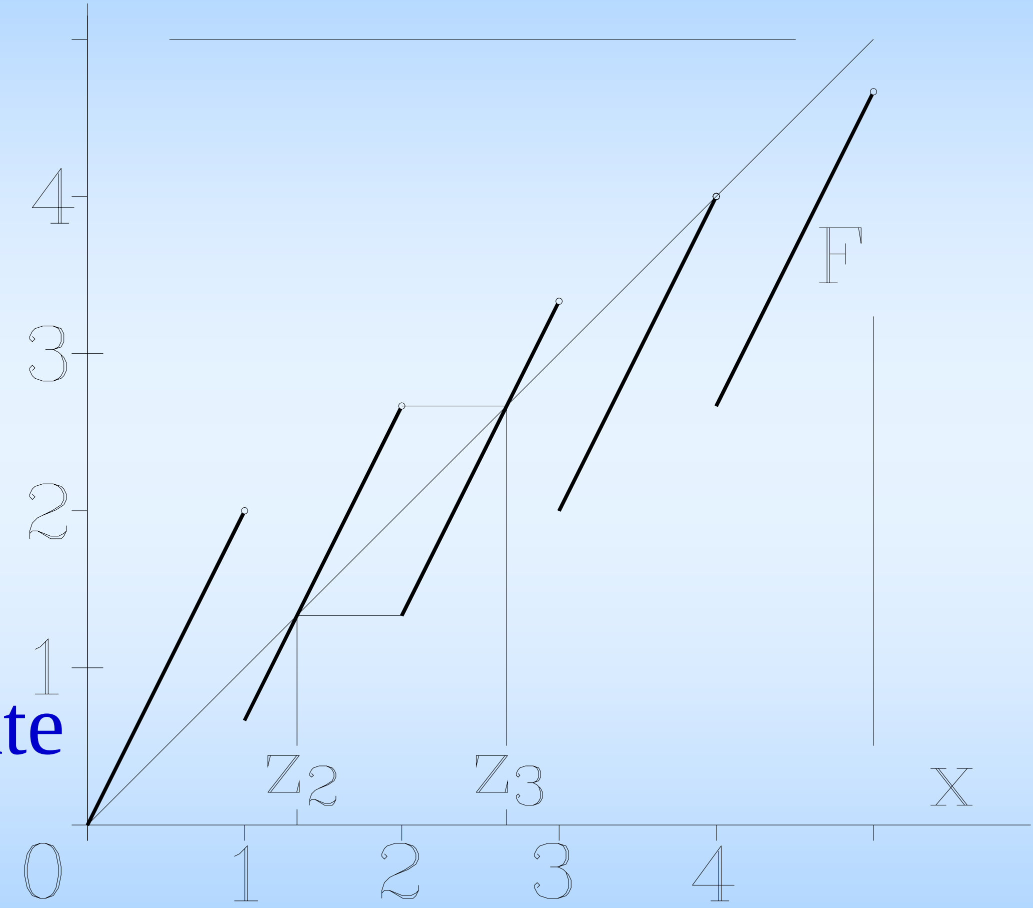


Typical in digitally
controlled machines,
caused partly by delay

Butterfly effect

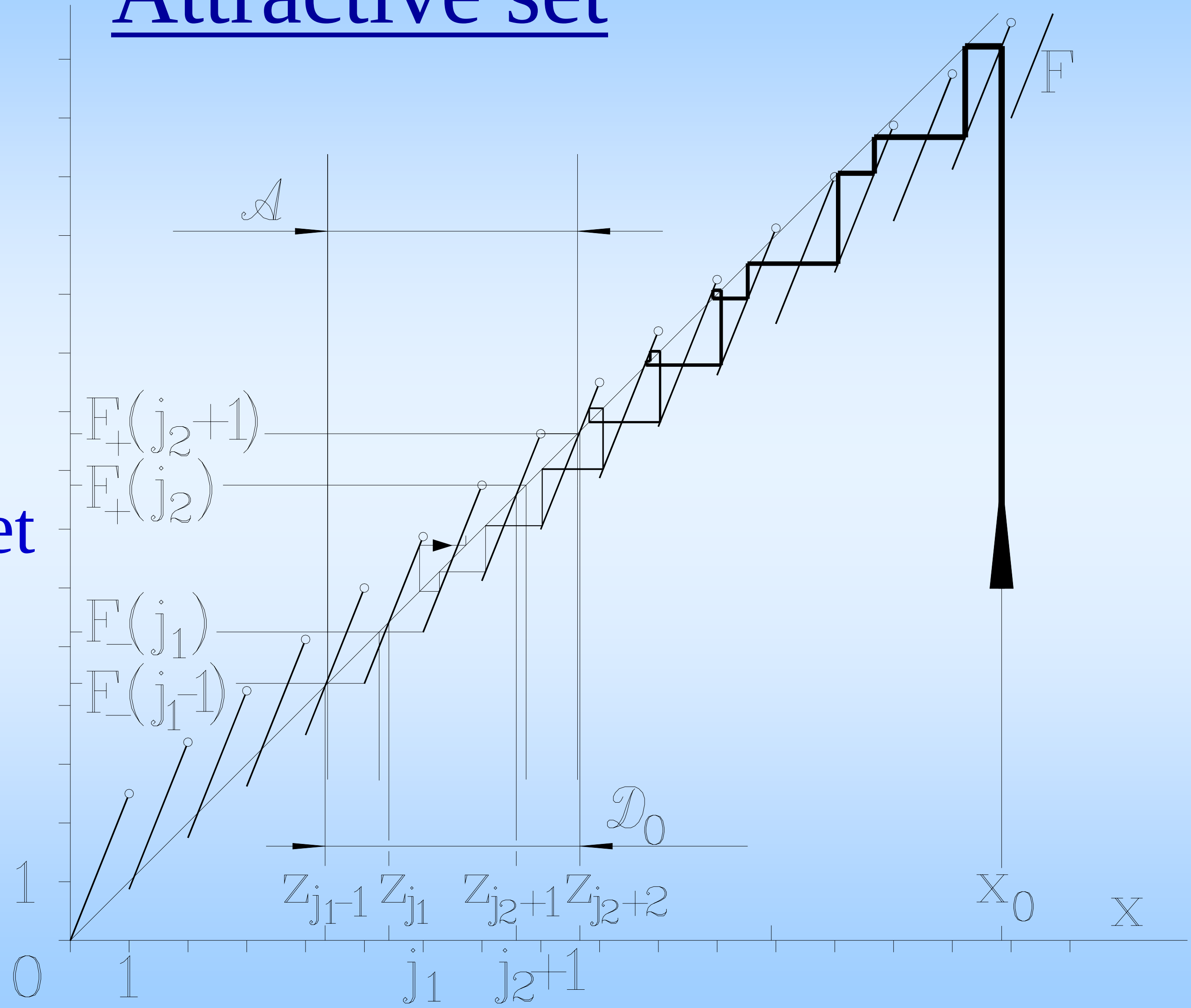
Prop. 1 The map has sensitive dependence on initial conditions

Horseshoe (Smale): invariant Cantor set on which the map is topologically conjugate to a Bernoulli shift on 2 symbols.

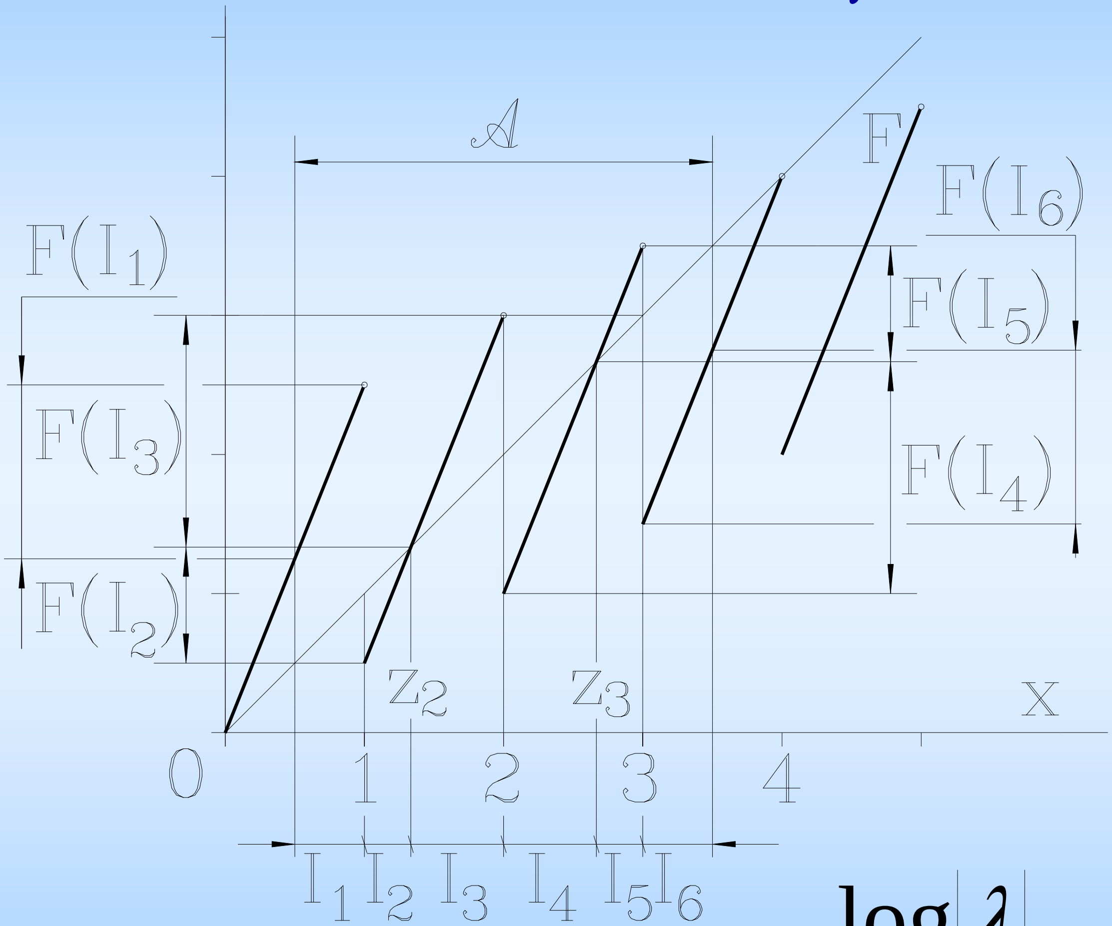


Attractive set

Prop. 2 ***A*** is a positively invariant attractive set



Characterization of μ-chaos (a=5/2, b=2)



$$\mathbf{A} = \begin{pmatrix} 0&0&1&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&1&1&0 \\ 0&1&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&1&0&0 \end{pmatrix}$$

Fractal dim.: $C(\Lambda) = \dfrac{\log|\lambda|_{\max}}{\log a} = \dfrac{\log 2.38}{\log 2.5} = 0.94$

Transient chaos

Unpredictable transient behavior of machines

The transient motion disappears “suddenly”

Exponential decay cannot be used

Life expectancy, kick-out number, escape rate, etc. can be defined

Examples: Lorenz repellor (Yorke, 1979), tethered satellites (Troger, 1998), shimmy, robotics, digital control, control of chaos…

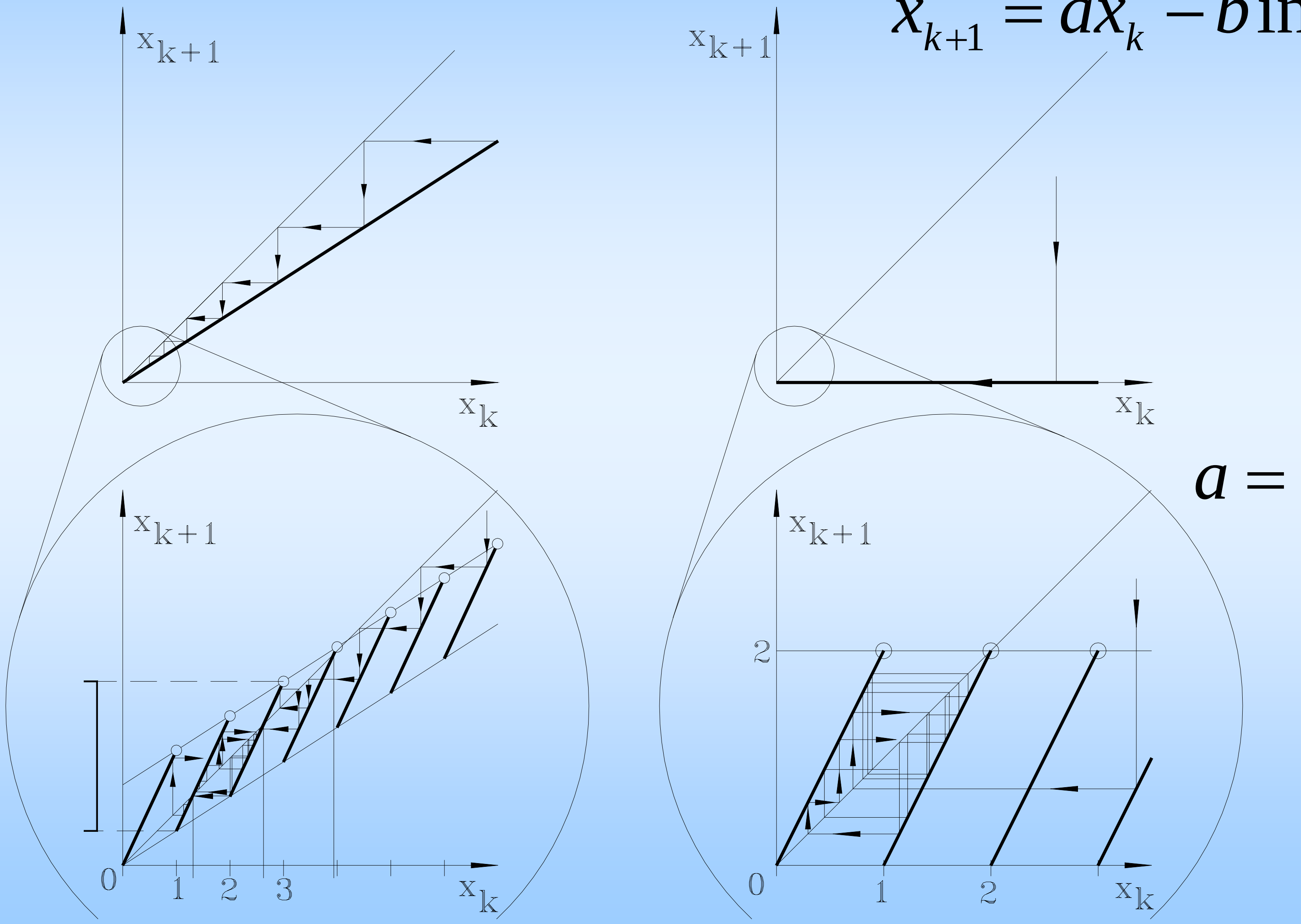
Examples from digital control

- PID control of machines in the presence of Coulomb friction
- Switch of robots from position control to force control, transient impacts with an elastic environment
- Stabilization of an unstable equilibrium or an unstable periodic motion of a machine (e.g.: balancing, control of chaos, …)

Trivial micro-chaos map

$$x_{k+1} = ax_k - b\,\mathrm{int}(x_k)$$

$$a = b = 2$$



Kick-out number

$a = b = 2$

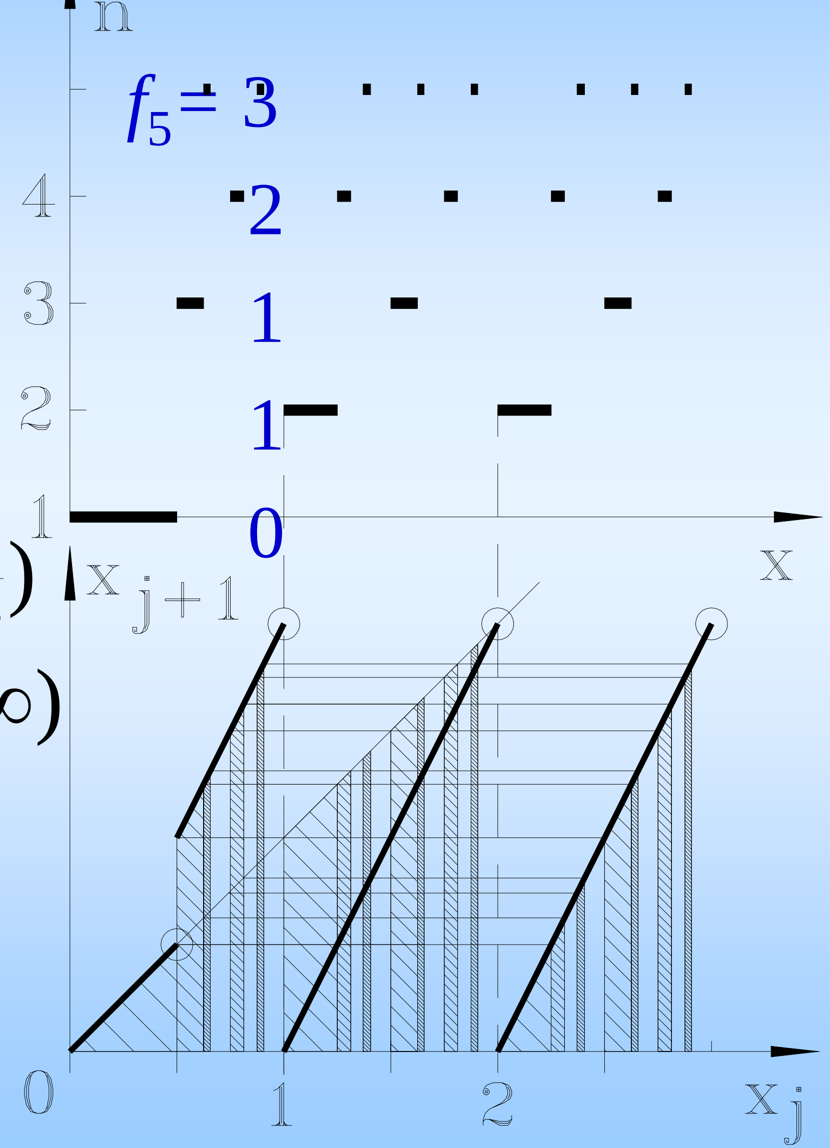
Fibonacci series: $f_n = f_{n\text{-}1} + f_{n\text{-}2}$

Length of intervals: $1/2^n$

$$x_j \in [1,2] \Rightarrow \sum_{n=1}^{\infty} \frac{f_n}{2^n} = 1$$

$$x_{j+1} = \begin{cases} x_j, & x_j \in [0, \frac{1}{2}) \\ 2x_j - 2\,\mathrm{int}(x_j), & x_j \in [\frac{1}{2}, \infty) \end{cases}$$

$$M = \lim_{N \to \infty} \frac{\int_0^N n\, \mathrm{d}x}{\int_0^N \mathrm{d}x} = \sum_{n=1}^{\infty} n \frac{f_n}{2^n} = 6$$

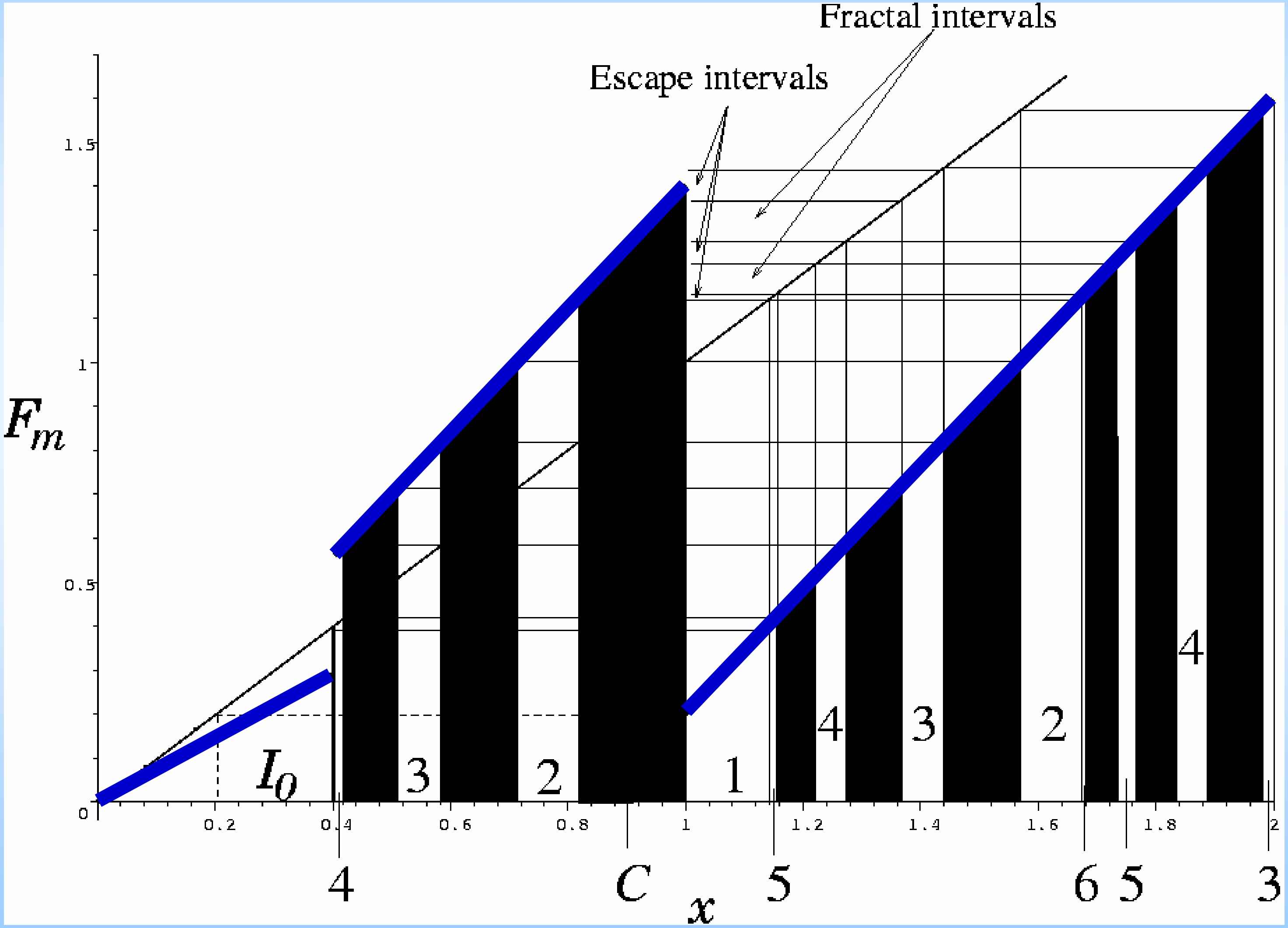


Non-trivial transient micro-chaos

a=1.4

b=1.2

I_0=0.2



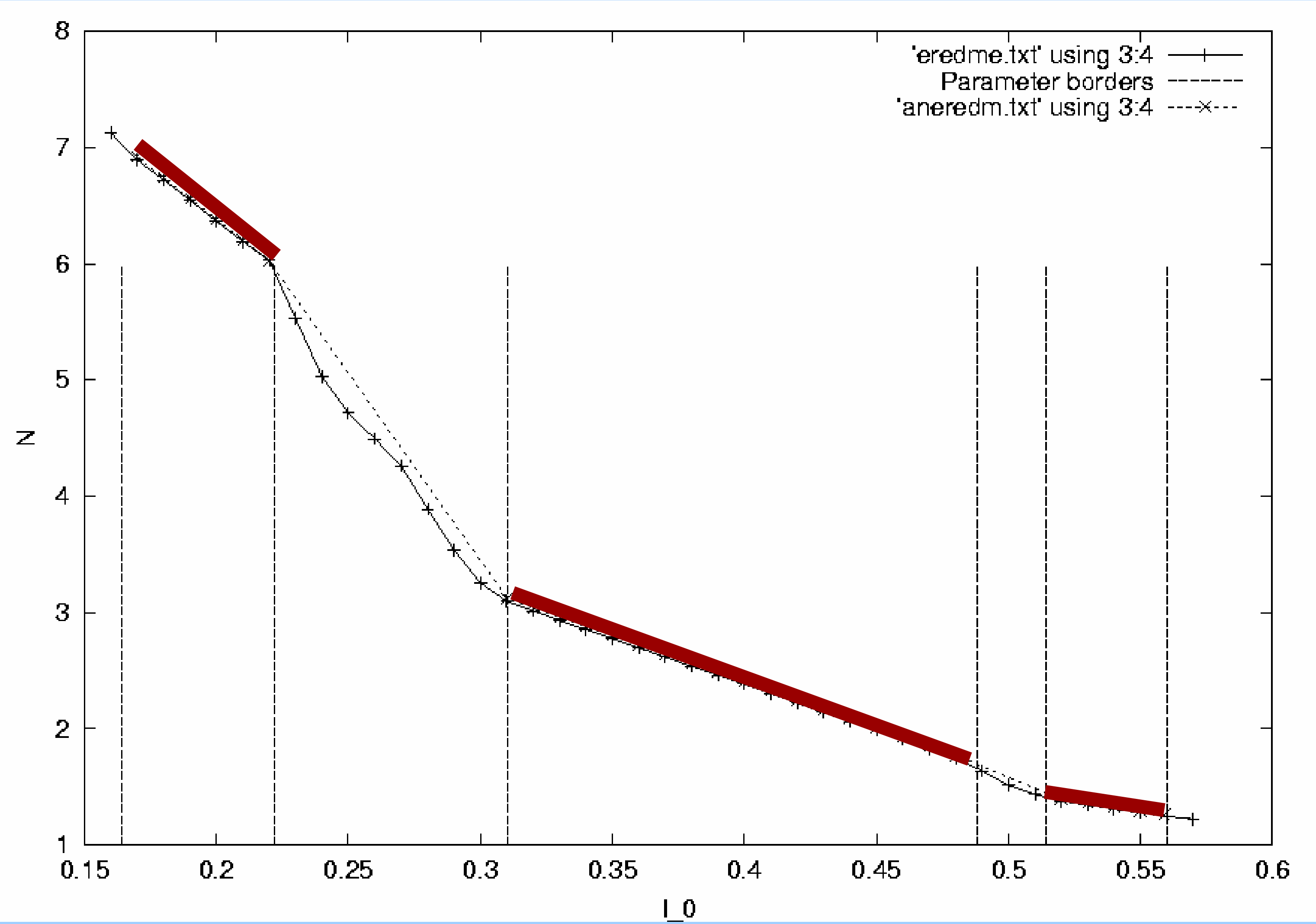
Non-trivial mean kick-out numbers

M

a=1.4

b=1.2

I_0=>



Robotic (dynamic) balancing

Even if vibration problems are all settled, there are still serious drawbacks:

- Balancing should be possible on any inclination, without knowing the exact vertical direction
- Balancing should work in space
- Balancing should incorporate gyroscopic effects

Study human balancing in more details!
elderly people, sportsmen –

delay and threshold