How delay equations arise in Engineering?

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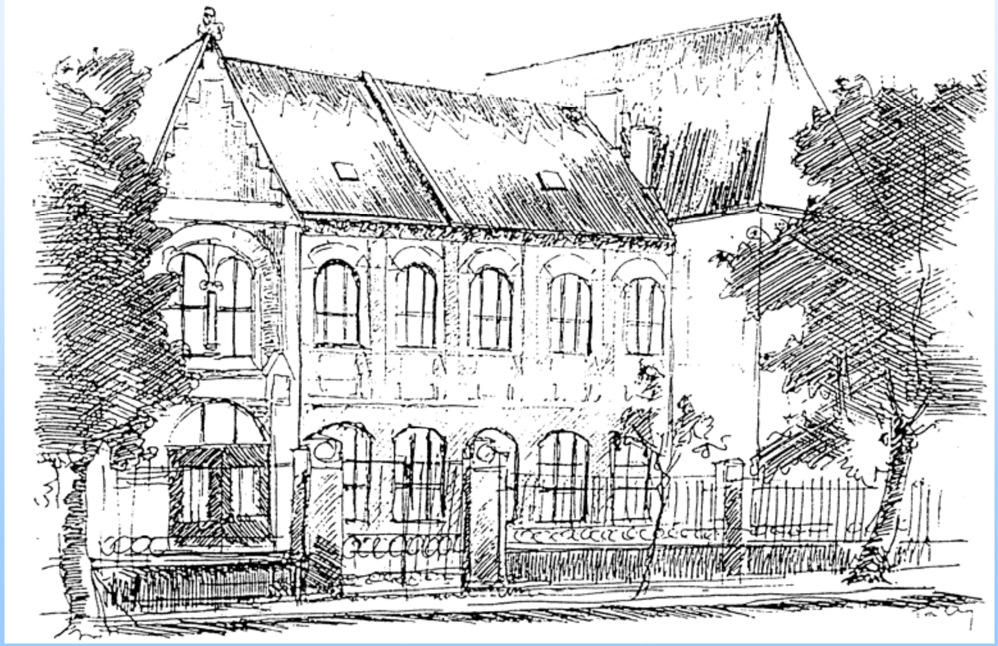


Contents

Answer: Delay equations arise in Engineering…

… by the information system (of control), and
by the contact of bodies.

- Linear stability & subcritical Hopf bifurcations
- Robotic position and force control
- Balancing – human and robotic
- **Contact problems**
  - Shimmying wheels (of trucks and motorcycles)
  - **Machine tool vibrations**

Motivation: Chatter

~ (high frequency) **machine tool vibration**

"… Chatter is the most obscure and delicate of all problems facing the machinist – probably ***no rules or formulae*** can be devised which will accurately guide the machinist in taking maximum cuts and speeds possible without producing chatter."

(Taylor, 1907).

(Moon, Feeny, 1996)

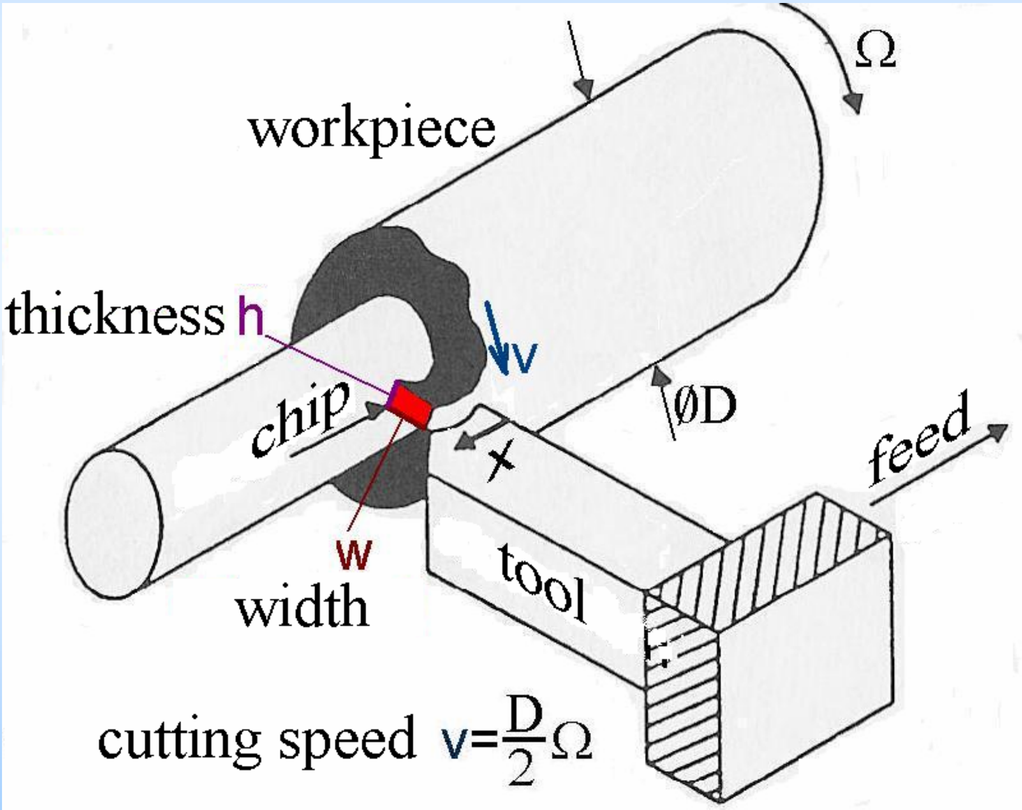
rezges-r.avi

Efficiency of cutting

Specific amount of material cut within a certain time

video

$$\dot{V} = wh\Omega\frac{D}{2}$$



where

w – chip width

h – chip thickness

Ω ~ cutting speed

Efficiency of cutting

Specific amount of material cut within a certain time

$$\dot{V} = wh\Omega\frac{D}{2}$$

where

w – chip width

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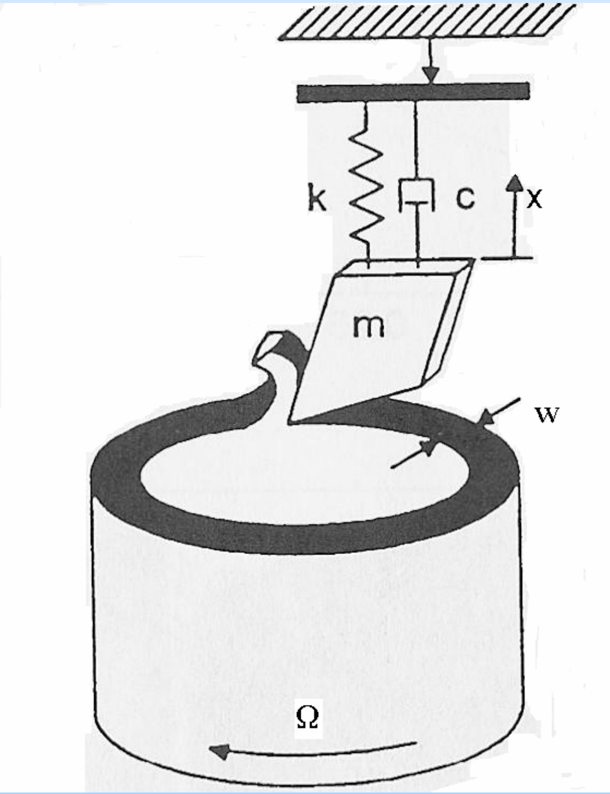
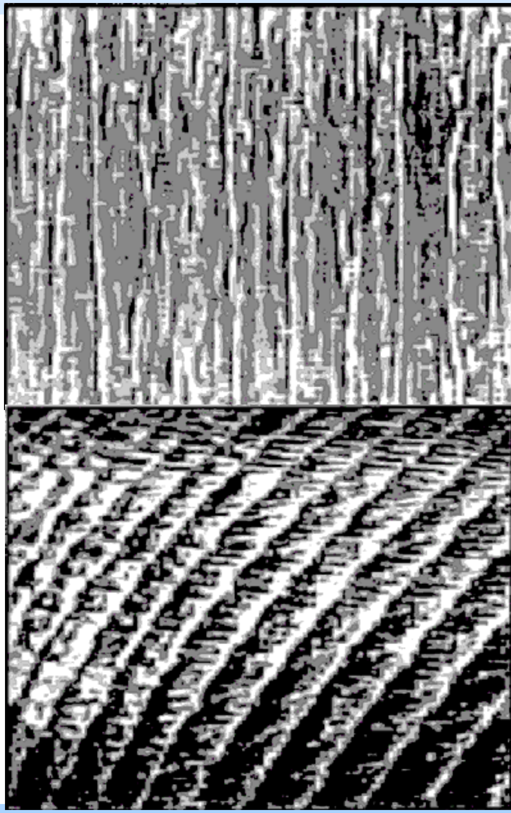
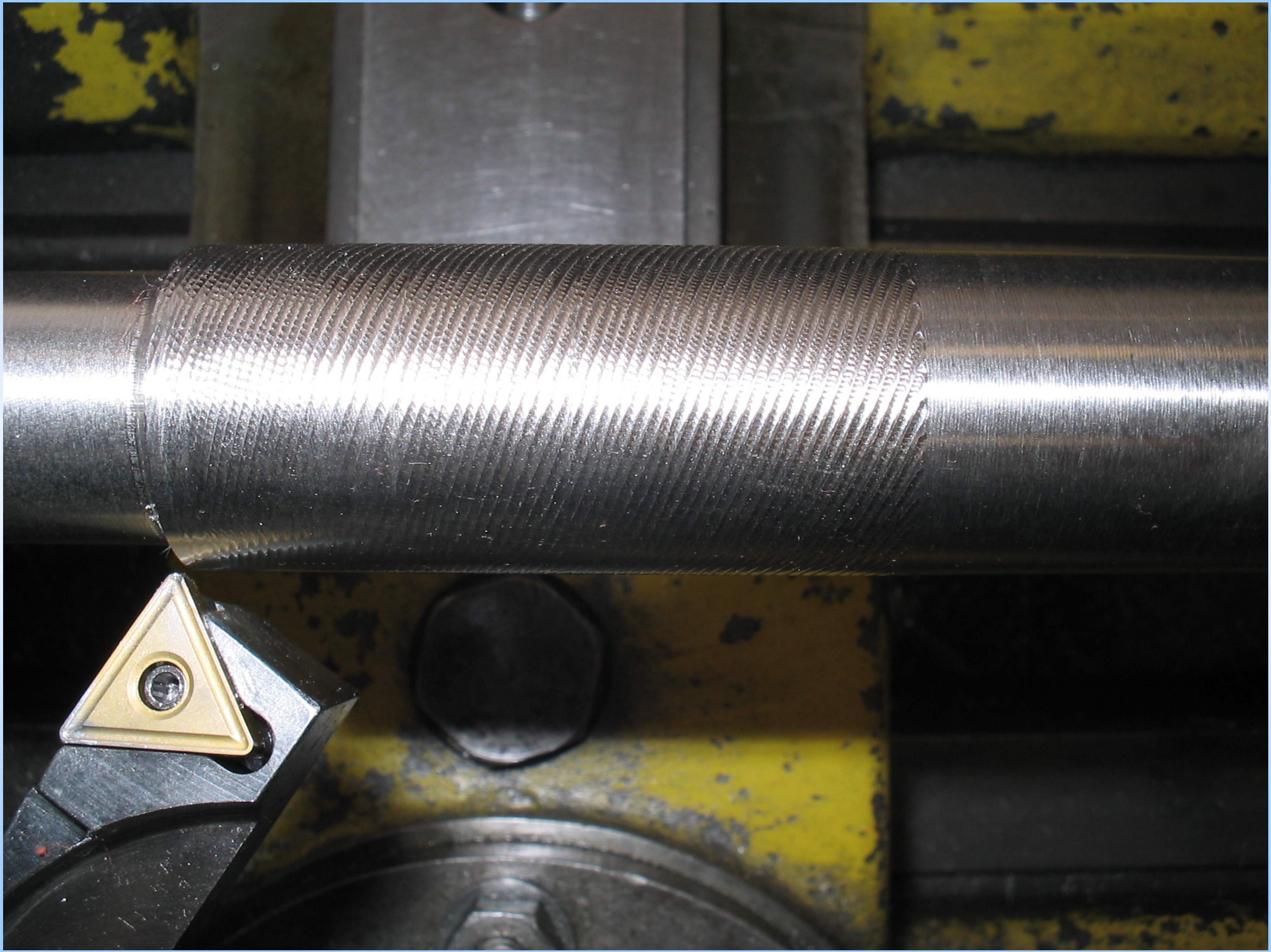




surface quality



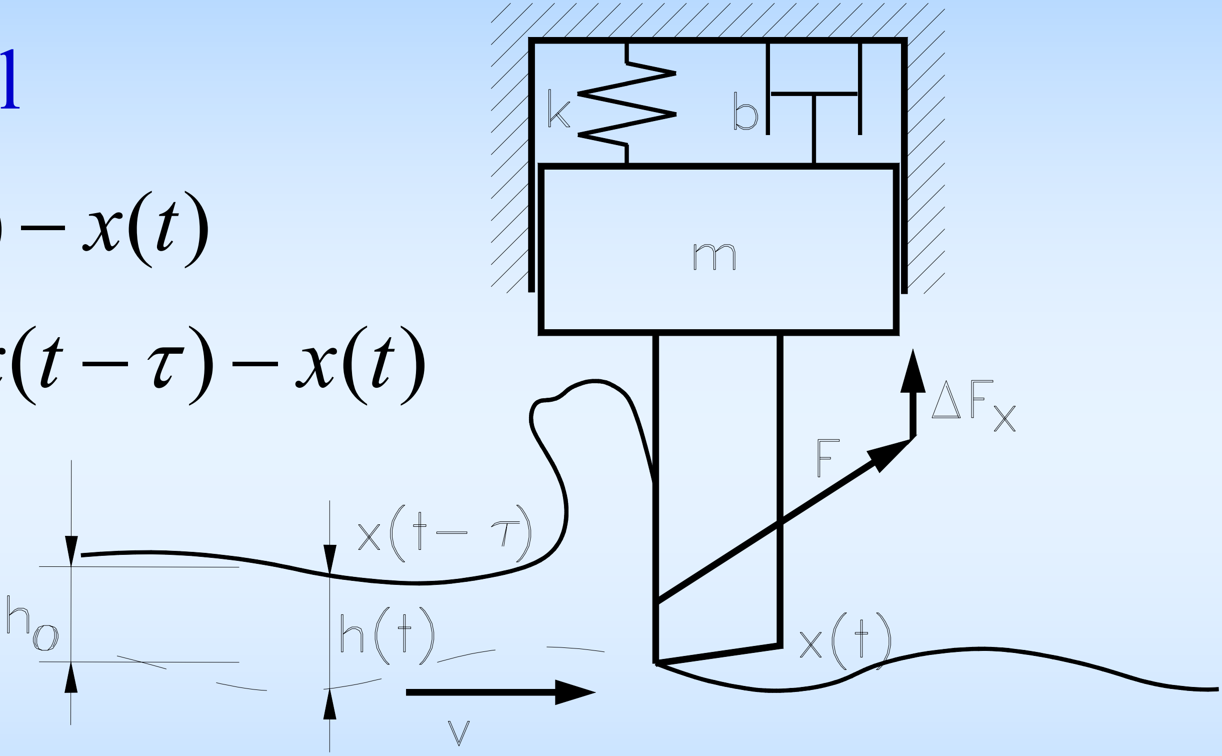
Modelling – regenerative effect

Mechanical model

$$h(t) = h_0 + x(t-\tau) - x(t)$$

$$\Delta h = h(t) - h_0 = x(t-\tau) - x(t)$$

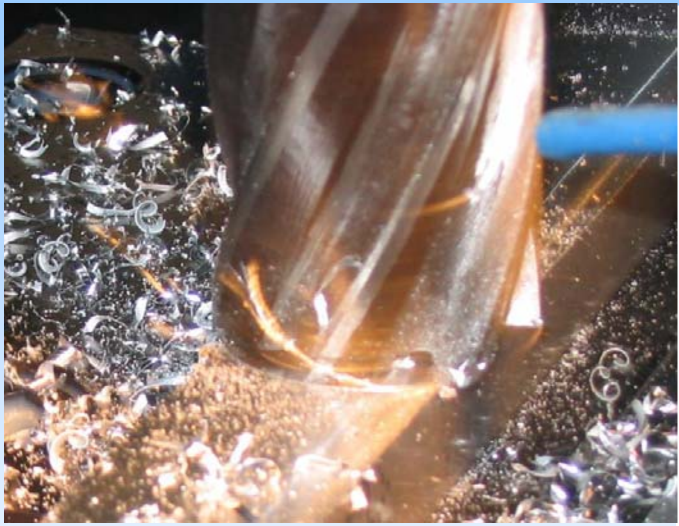
τ – time period of revolution

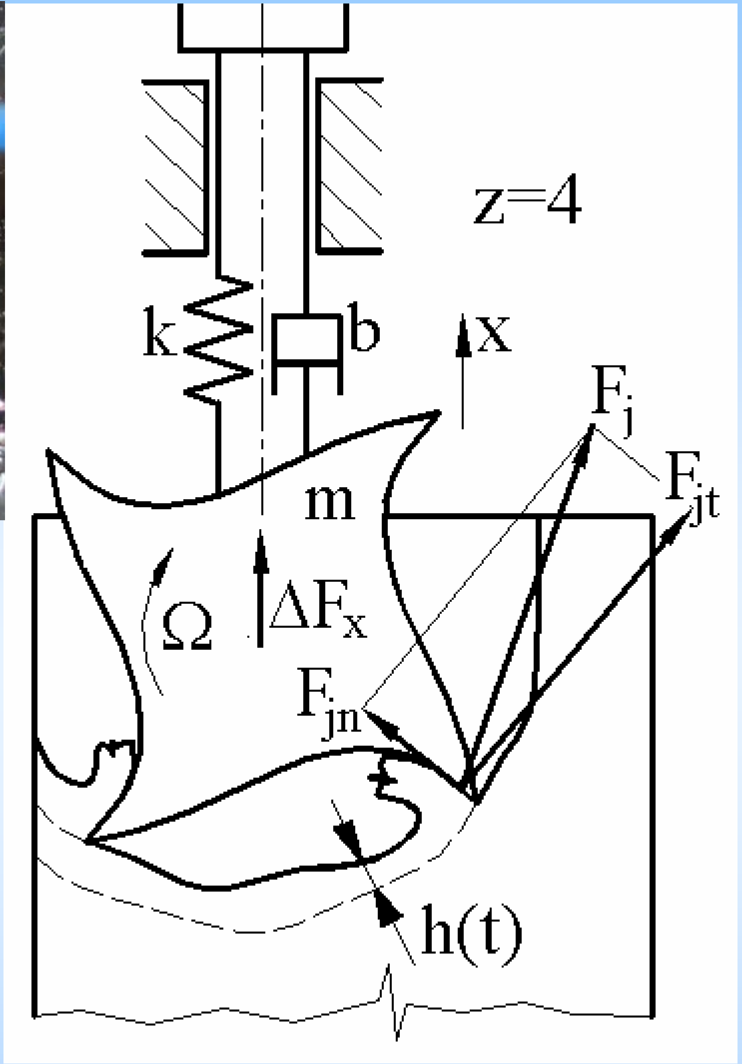


Mathematical model

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}\Delta F_x(\Delta h)$$

Milling





Mechanical model:

- number of cutting edges in contact varies *periodically* with period equal to the delay

$$\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+(\omega_n^2+\frac{k_1(t)}{m})x(t)=\frac{k_1(t)}{m}x(t-\tau)$$

$$k_1(t+\tau)=k_1(t)$$

Stabilizing inverted pendula

Stephenson (1908): periodically forced pendulum

$$(m\,l_S^{\,2}+\Theta_S)\,\ddot{\varphi}+(\boxed{-m\,g\,l_S}+\boxed{m\,r\,\omega^2\,l_S\cos(\omega\,t)})\,\varphi=0$$

(M Levi)

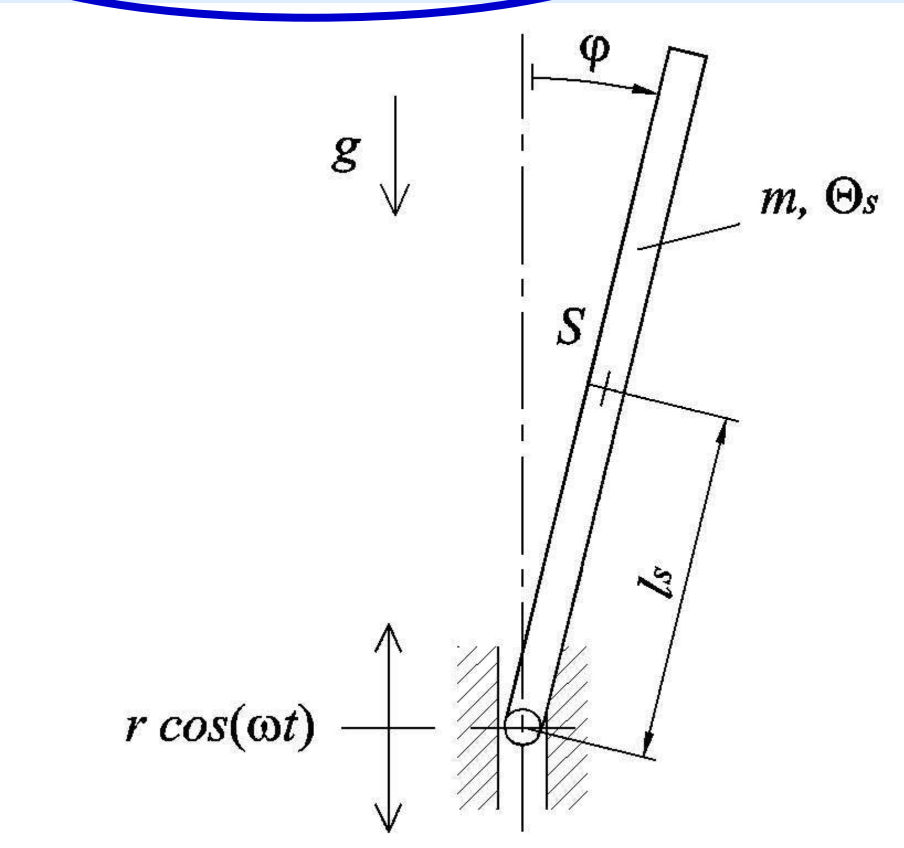
Mathematical background:

Mathieu equation (1868)

$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=0$$

$x=0$ can be stable in

Ljapunov sense for $\delta<0$



Stabilization with parametric excitation

Stabilization with parametric excitation

Fingers in shaken cornstarch solution
Univ Texas at Austin

cornstarch

(Deegan, Merkt, Swinney, 2004)

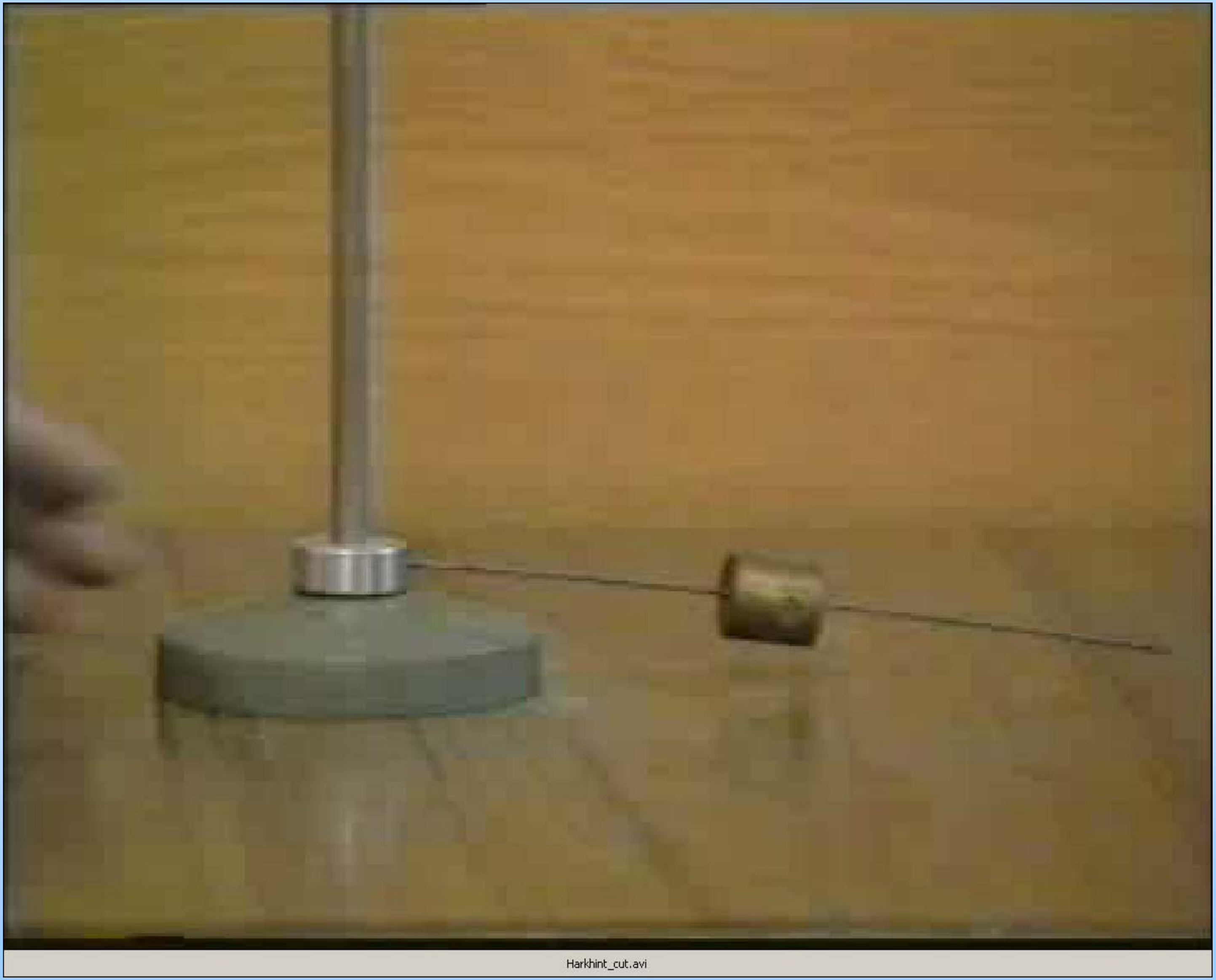
Ship stabilization

video1

(Zelei, Stepan, 2005)

video2

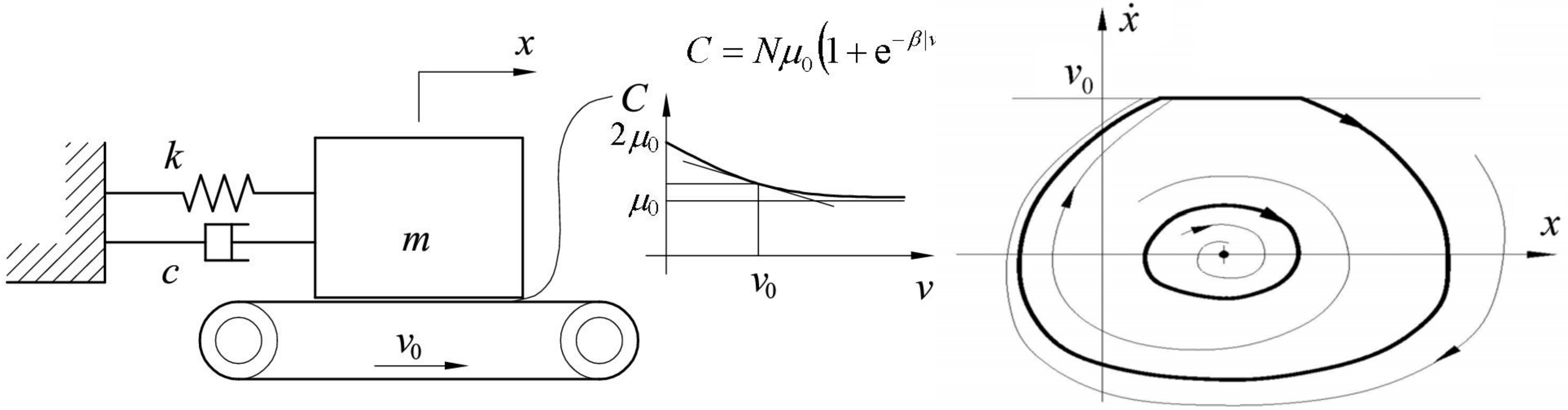
Instability – period-2 oscillations (swing)



Stick&slip – unstable periodic motion

Experiments with brakepad-like arrangements
(R Horváth, Budapest / Auburn)

slip



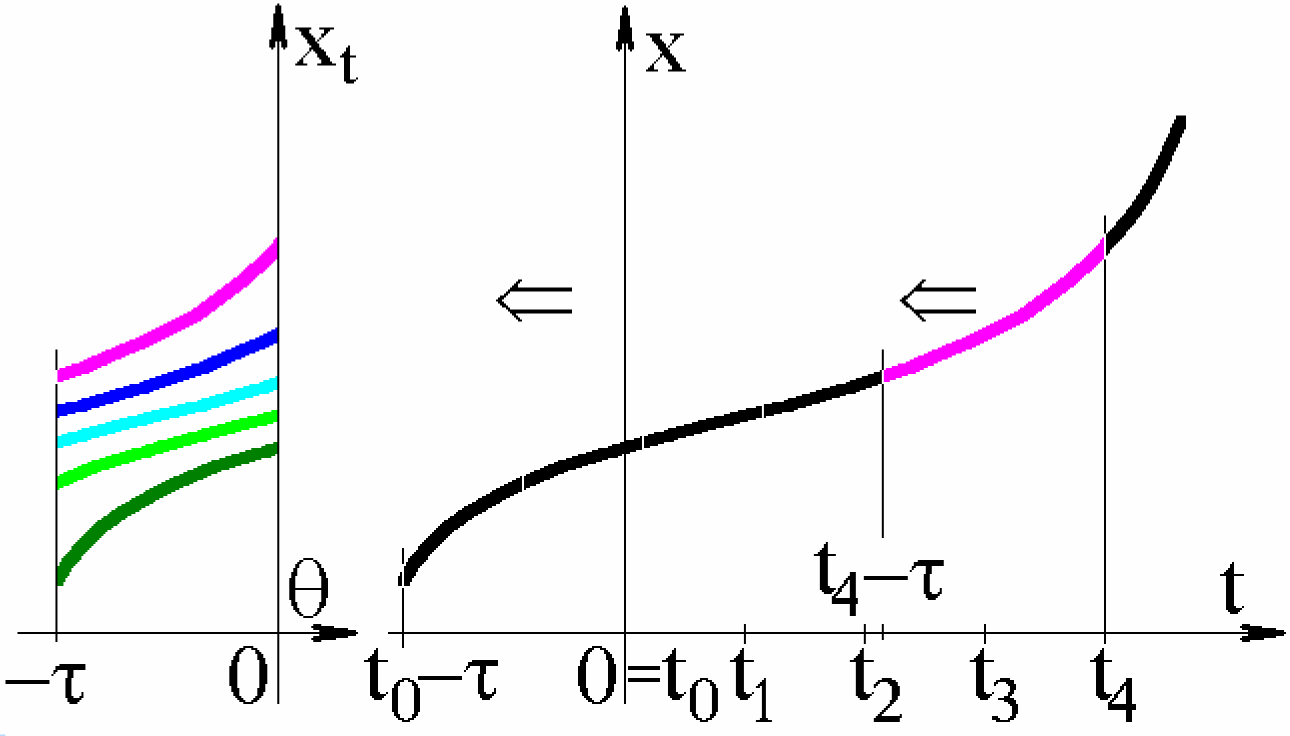
Leine, van Campen, van de Vrande (2000)

Delay Diff Equ (DDE) – Functional DE

Time delay & infinite dimensional phase space:

$$x_t(\vartheta) = x(t+\vartheta)$$

$$\vartheta \in [-\tau, 0]$$



$$\dot{x}(t) = Lx_t$$

$$\dot{x}(t) = \int_{-\tau}^{0} x(t+\vartheta)\,\mathrm{d}\eta(\vartheta)$$ Riesz Representation Theorem

‘60 ‘70 ‘90

Floquet for FDE’s – Halanay,Hale,Lunel

$$\dot{\mathbf{x}}(t)=\mathbf{L}(t,\mathbf{x}_t), \quad \mathbf{L}(t+T,\mathbf{x}_t)=\mathbf{L}(t,\mathbf{x}_t), \ \ \mathbf{x}_t=\mathbf{U}(t)\mathbf{x}_0 \Rightarrow \mathbf{U}(T)$$

Characteristic equation: $\mathrm{Ker}(\mu\mathbf{I}-\mathbf{U}(T))\neq\{0\}$

μ - characteristic multiplier $\mu=\mathrm{e}^{\lambda T}$

λ - characteristic exponent $|\mu|<1 \Leftrightarrow \ \mathrm{Re}\,\lambda<0$

Th1. $x = 0$ is asymptotically stable, if and only if all the (infinite number of) characteristic multipliers are in modulus less than one.

Th2. $\mu=\exp(\lambda T)$ is a characteristic multiplier if and only if, there exists a nontrivial solution in the form $y(t)=p(t)\exp(\lambda t)$, where $p(t)=p(t+T)$.

Stability of linear RFDEs of *n* DoF systems

Delayed mechanical systems include 2nd derivatives:

$$M\ddot{x}(t)+\int_{-h}^{0}\mathrm{d}_{\vartheta}B(t,\vartheta)\dot{x}(t+\vartheta)+\int_{-h}^{0}\mathrm{d}_{\vartheta}K(t,\vartheta)x(t+\vartheta)=0$$

Autonomous systems: $B(t,\vartheta)\equiv B(\vartheta),\ K(t,\vartheta)\equiv K(\vartheta)$

Trial solution: $\boxed{x(t)=A\mathrm{e}^{\lambda t}}$ $A\in R^{n}$

Characteristic roots: Re $\lambda_j<0$, j=1,2,… ⇔ stability

$$\boxed{D(\lambda)=\det(M\lambda^{2}+\int_{-h}^{0}\lambda\mathrm{e}^{\lambda\vartheta}\mathrm{d}B(\vartheta)+\int_{-h}^{0}\mathrm{e}^{\lambda\vartheta}\mathrm{d}K(\vartheta))}$$

D-curves: $R(\omega)=\operatorname{Re}D(i\omega),\ S(\omega)=\operatorname{Im}D(i\omega),\ \omega\in[0,\infty)$

Selecting the stable regions among them…(?!)

Non-autonomous linear RFDEs

$$M\ddot{x}(t)+\int_{-h}^{0}\mathrm{d}_{\vartheta}B(t,\vartheta)\dot{x}(t+\vartheta)+\int_{-h}^{0}\mathrm{d}_{\vartheta}K(t,\vartheta)x(t+\vartheta)=0$$

Time-periodic systems: $B(t+T,\vartheta)=B(t,\vartheta)$

Trial solution: $\boxed{x(t)=p(t)\mathrm{e}^{\lambda t}}$ $K(t+T,\vartheta)=K(t,\vartheta)$

$$p(t+T)=p(t)=\sum_{k=0}^{+\infty}\left(A_k\cos(k\tfrac{2\pi}{T}t)+B_k\sin(k\tfrac{2\pi}{T}t)\right)$$

Hill's infinite dimensional determinant (1886) ⇒

characteristic function ⇒ characteristic roots λ

Re $\lambda_j<0$, j=1,2,… ⇔ stability ⇔ $|\mu_j|<1$, j=1,2,…

for characteristic multipliers $\mu=\mathrm{e}^{\lambda T}$ of fund. op. at T

The delayed Mathieu equation

Analytically constructed stability chart for testing numerical methods and algorithms

$$\boxed{\ddot{x}(t) + \kappa\dot{x}(t) + (\delta + \varepsilon\cos t\,)x(t) = b\,x(t-2\pi)}$$

Time delay and time periodicity are equal:

$$T = \tau = 2\pi$$

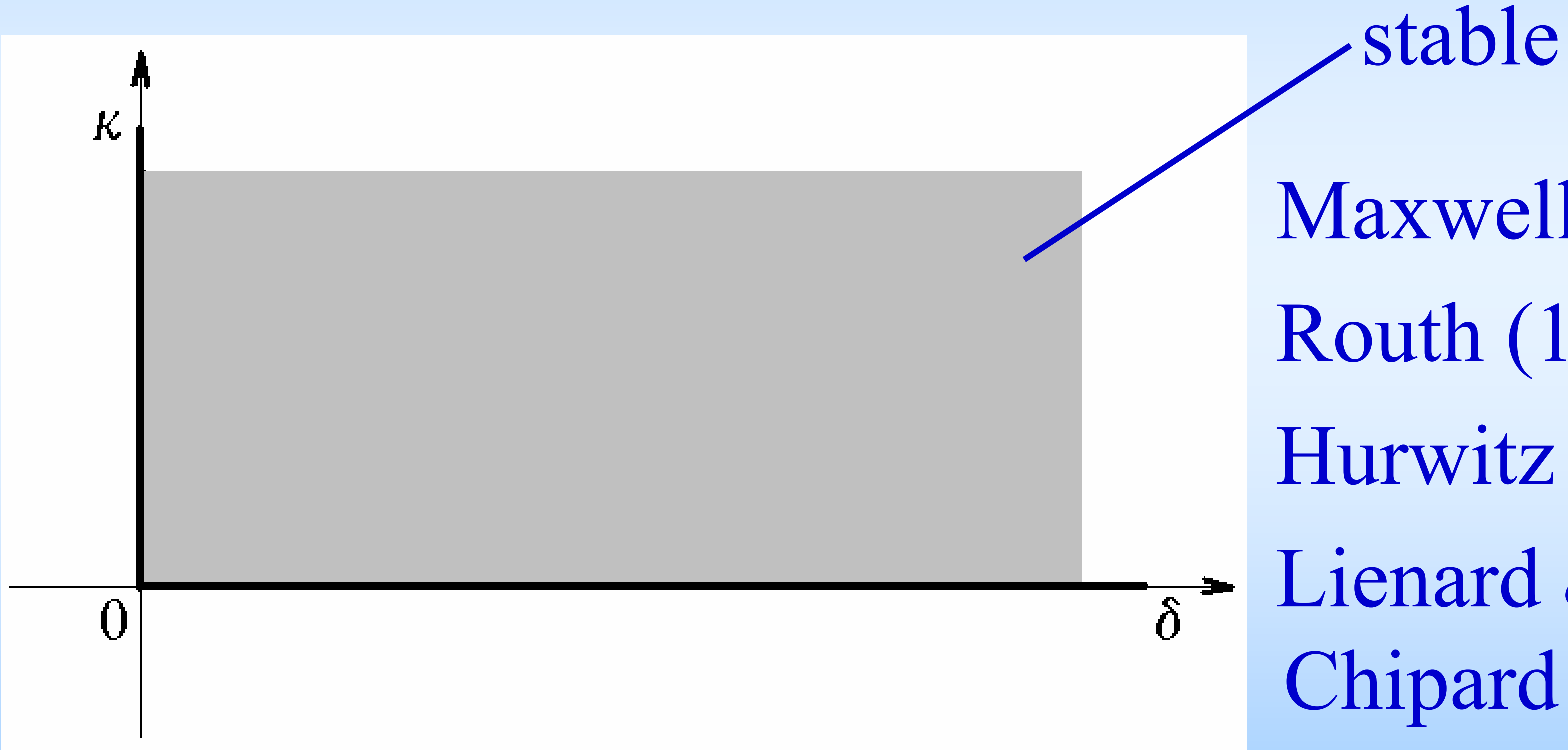
$b = \varepsilon = 0$ Damped oscillator

$b = 0$ Mathieu equation (1868)

$\varepsilon = 0$ Delayed oscillator (1941 – shimmy)

The damped oscillator

$$\ddot{x}(t) + \kappa\,\dot{x}(t) + \delta\,x(t) = 0$$



Maxwell(1865)

Routh (1877)

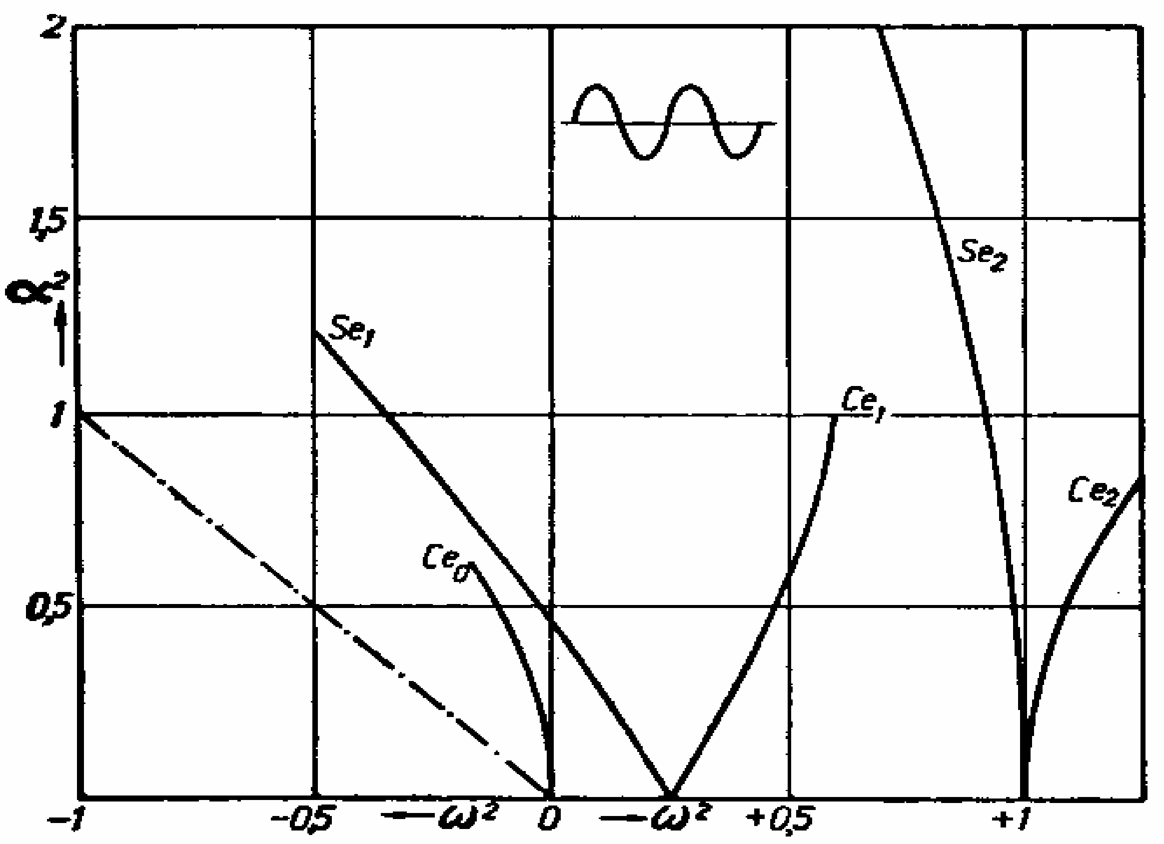
Hurwitz (1895)

Lienard &
Chipard (1917)

Stability chart – Mathieu equation

$$\ddot{x}(t)+(\delta+\varepsilon\cos t)x(t)=0$$

Fig. 6.



Floquet (1883)

Hill (1886)

Rayleigh(1887)

van der Pol &

Strutt (1928)

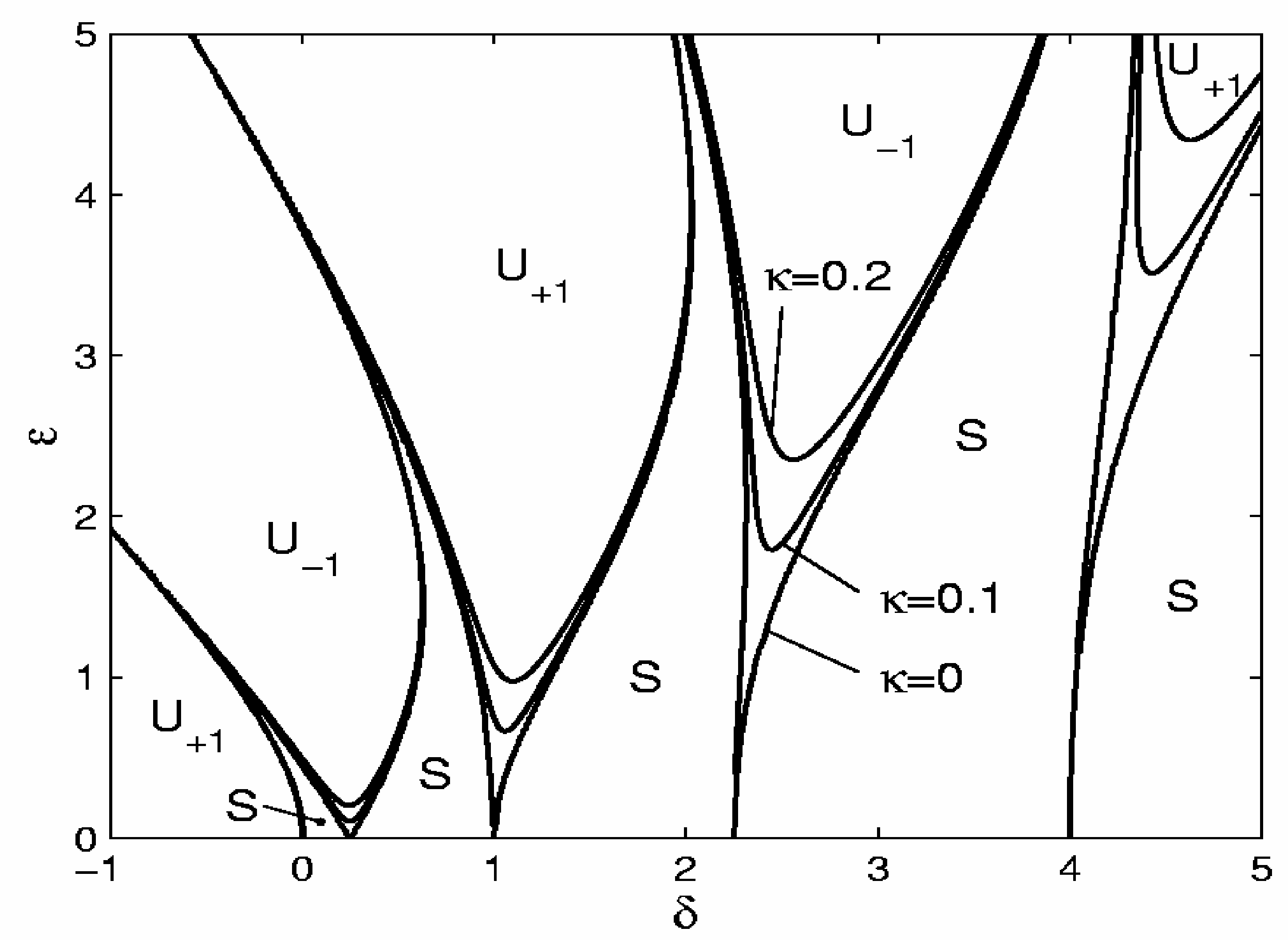
Sinha (1992)

~swing(2000BC)

ulum (1908)

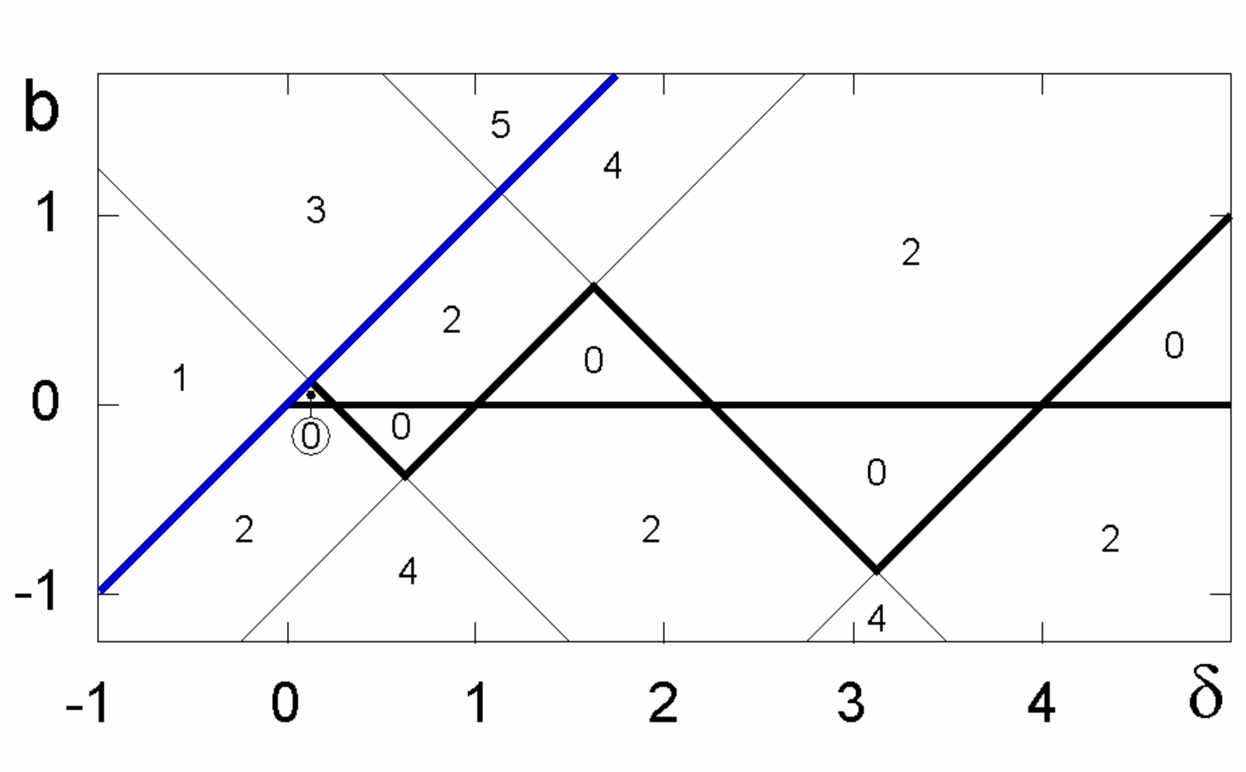
The damped Mathieu equation

$$\ddot{x}(t)+\kappa\,\dot{x}(t)+(\delta+\varepsilon\cos t\,)x(t)=0$$



The delayed oscillator

$$\ddot{x}(t) + \delta\, x(t) = b x(t - 2\pi)$$



Pontryagin (1942)

Nyquist (1949)

Bellman & Cooke (1963)

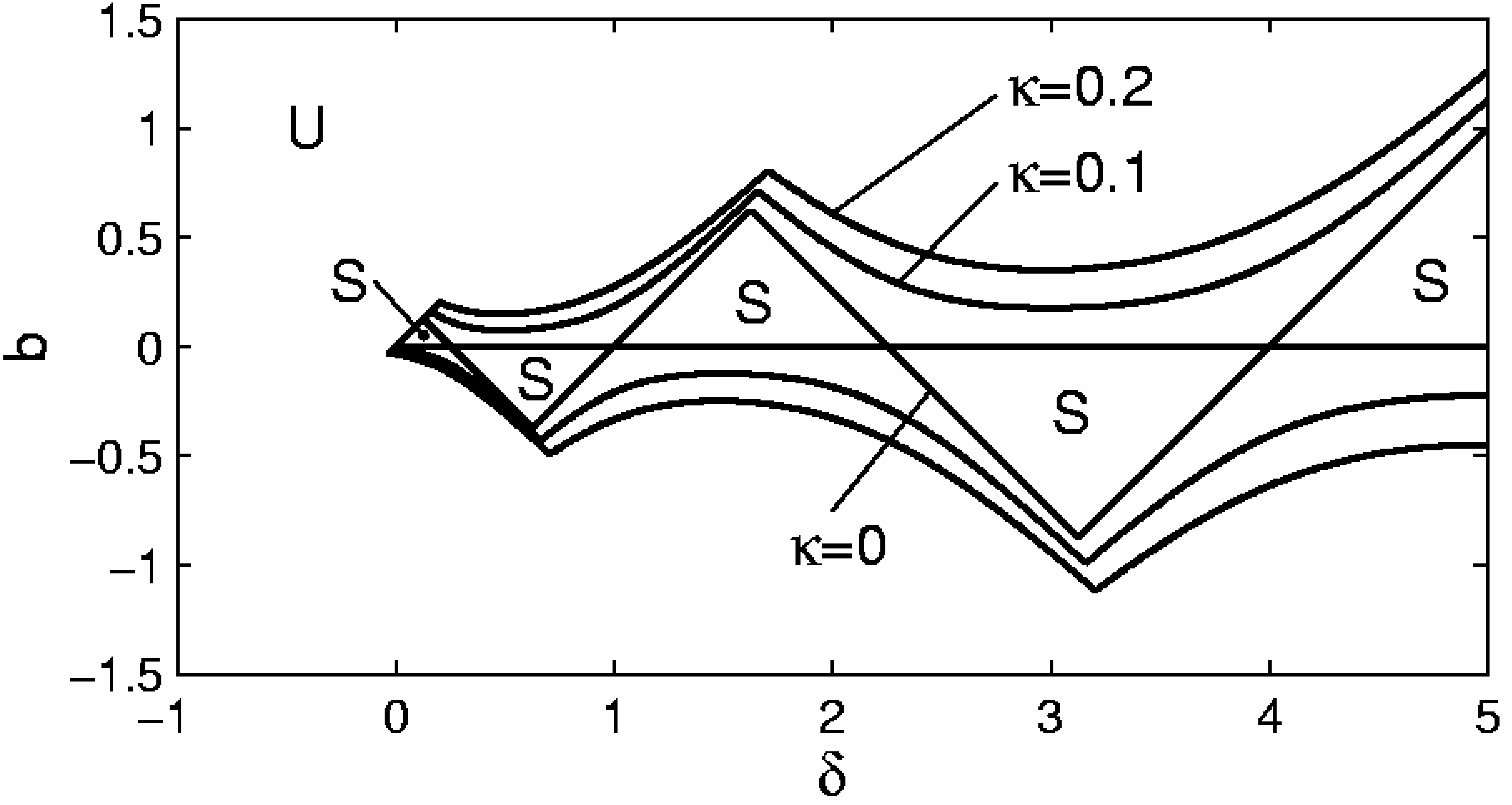
Kolmanovskii, Niculescu, Olgac

Hsu & Bhatt (1966)

(Stépán: Retarded Dynamical Systems, 1989)

Delayed oscillator with damping

$$\ddot{x}(t)+\kappa\dot{x}(t)+\delta x(t)=bx(t-2\pi)$$



The delayed Mathieu equation

$$\ddot{x}(t)+(\delta+\varepsilon\cos t\,)x(t)=b\,x(t-2\pi)$$

$$x(t)=\sum_{k=0}^{\infty}\left(A_k\,\mathrm{e}^{\mathrm{i}kt}+B_k\,\mathrm{e}^{-\mathrm{i}kt}\right)\mathrm{e}^{\lambda t}+\sum_{k=0}^{\infty}\left(\overline{A}_k\,\mathrm{e}^{-\mathrm{i}kt}+\overline{B}_k\,\mathrm{e}^{\mathrm{i}kt}\right)\mathrm{e}^{\overline{\lambda}t}$$

$$x(t)=\sum_{k=-\infty}^{\infty}C_k\,\mathrm{e}^{(\lambda+\mathrm{i}k)t}+\overline{C}_k\,\mathrm{e}^{(\overline{\lambda}-\mathrm{i}k)t}$$

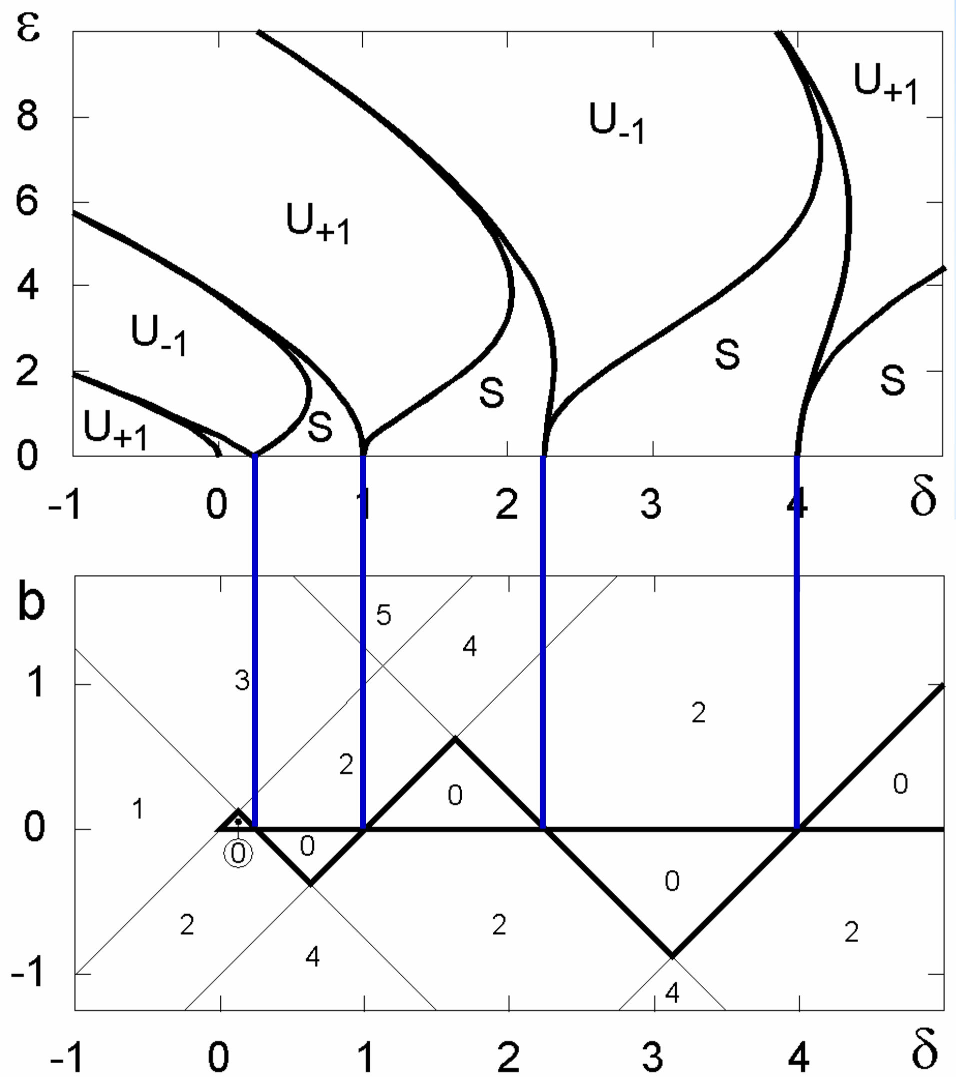
Harmonic balance

⇒ Hill’s determinant

$$\det\begin{pmatrix} \ddots & & \ddots \qquad\qquad \ddots & 0 & 0\ldots \\ \ldots 0 & \dfrac{\varepsilon}{2} & \delta+(\lambda+\mathrm{i}k)^2-b\,\mathrm{e}^{-2\pi(\lambda+\mathrm{i}k)} & \dfrac{\varepsilon}{2} & 0\ldots \\ \ldots 0 & 0 & \ddots \qquad\qquad \ddots & & \ddots \end{pmatrix}=0$$

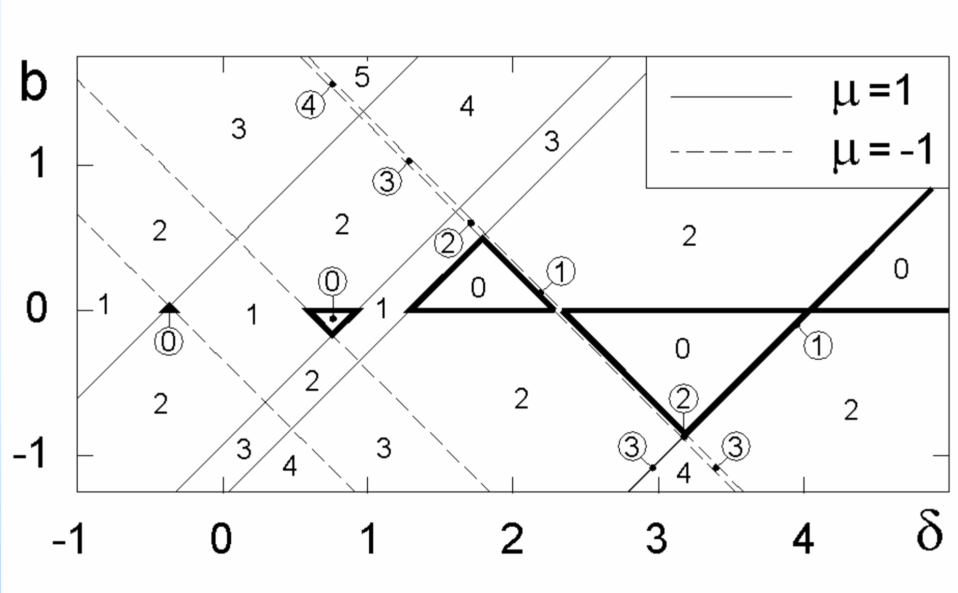
The delayed Mathieu – stability charts

$$\ddot{x}(t)+(\delta+\varepsilon\cos t\,)x(t)=b\,x(t-2\pi)$$



b=0

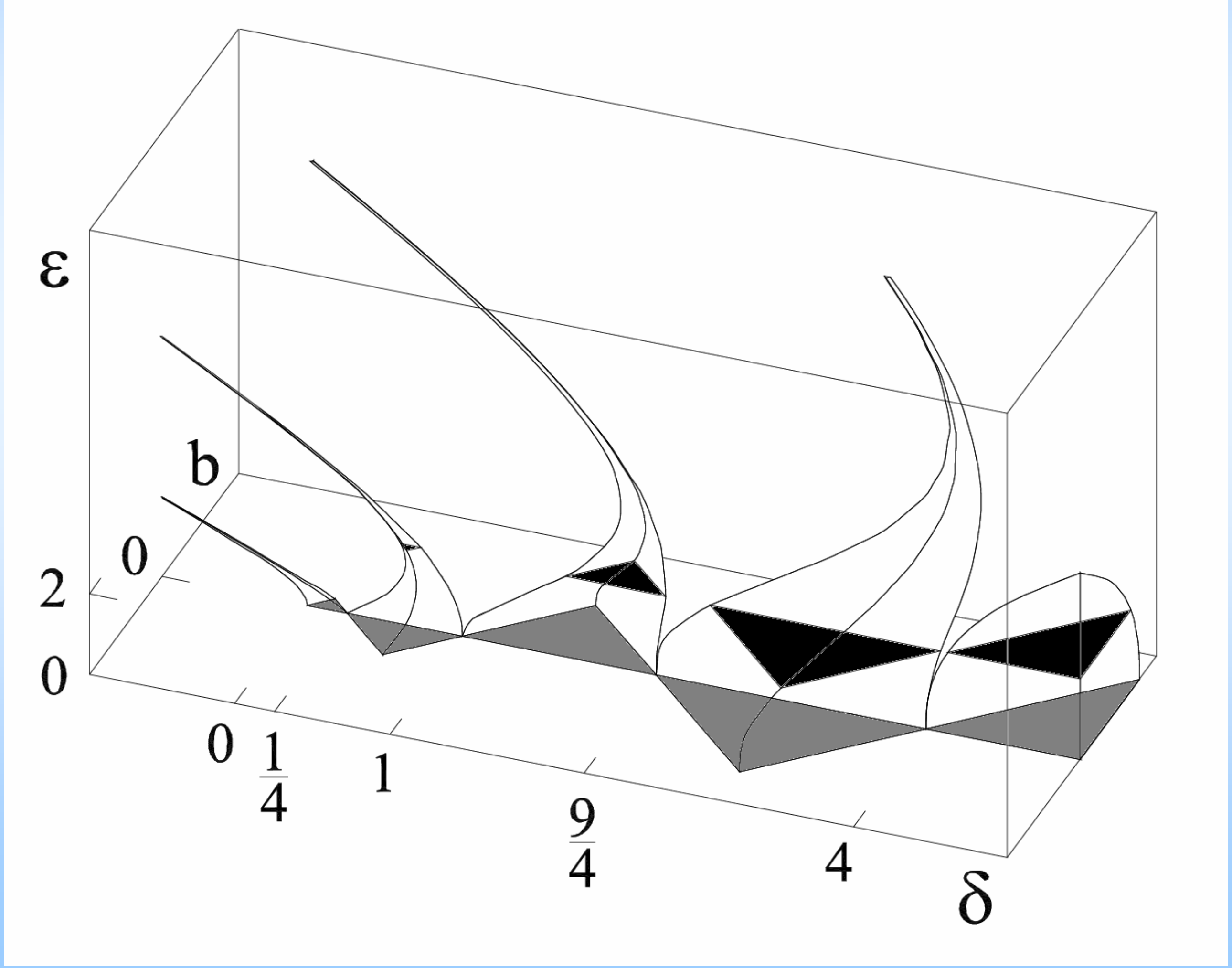
ε=0



ε=1

Stability chart of delayed Mathieu

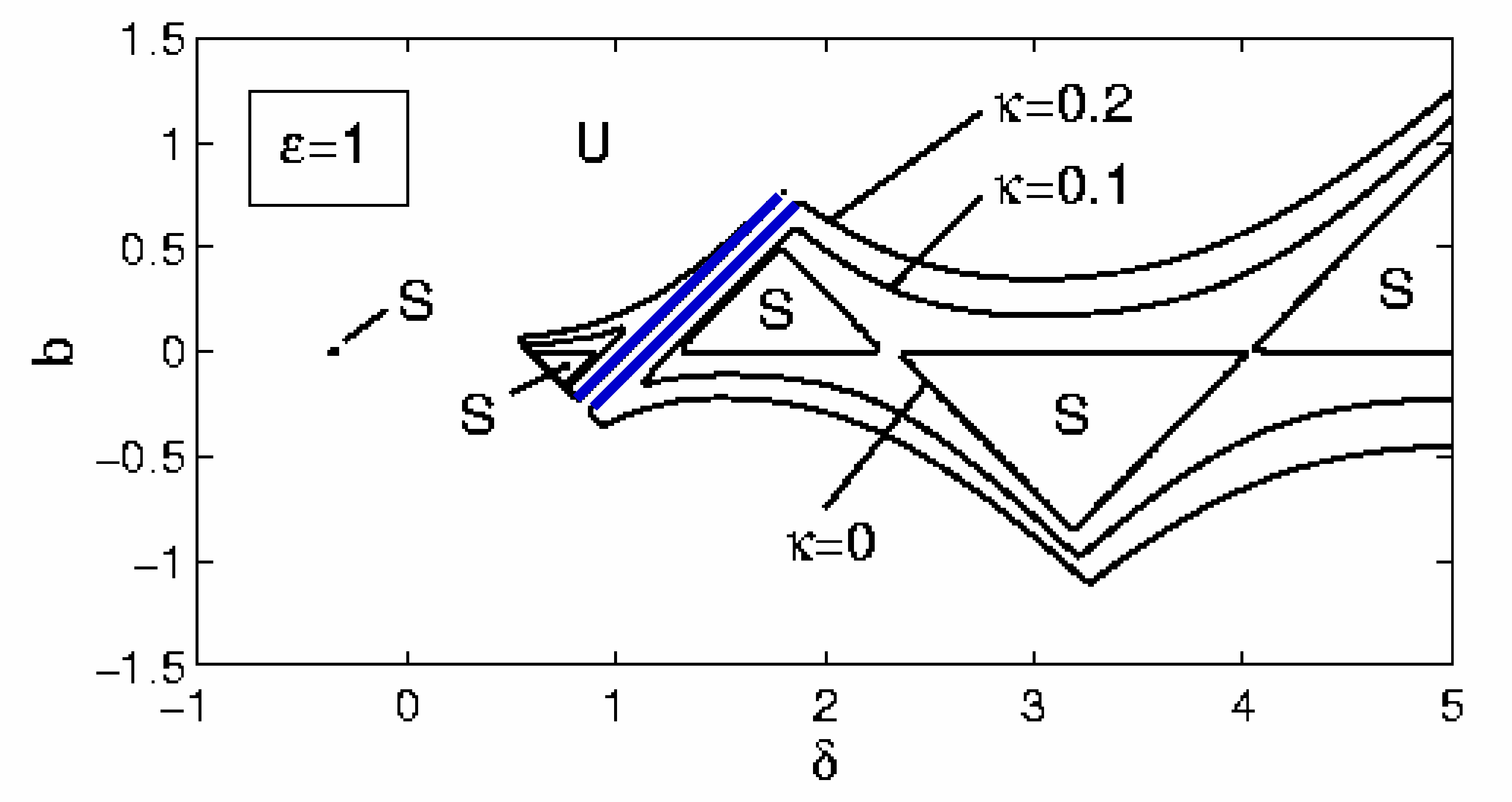
$$\ddot{x}(t) + (\delta + \varepsilon \cos t)x(t) = b\,x(t - 2\pi)$$



Insperger, Stépán (2002)

Test of damped delayed Mathieu equ.

$$\ddot{x}(t)+\kappa\dot{x}(t)+(\delta+\varepsilon\cos t)x(t)=b\,x(t-2\pi)$$



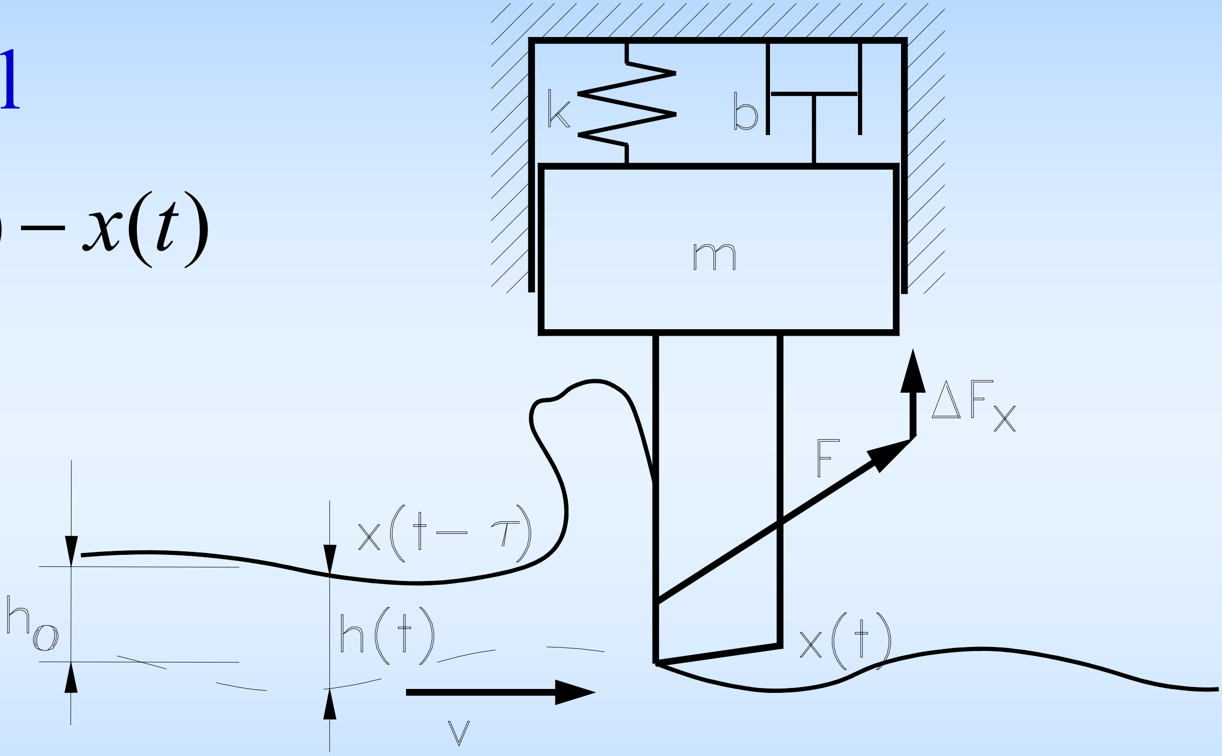
Modelling – regenerative effect

Mechanical model

$$h(t) = h_0 + x(t-\tau) - x(t)$$

$$\Delta h = h(t) - h_0$$

τ – time period of revolution

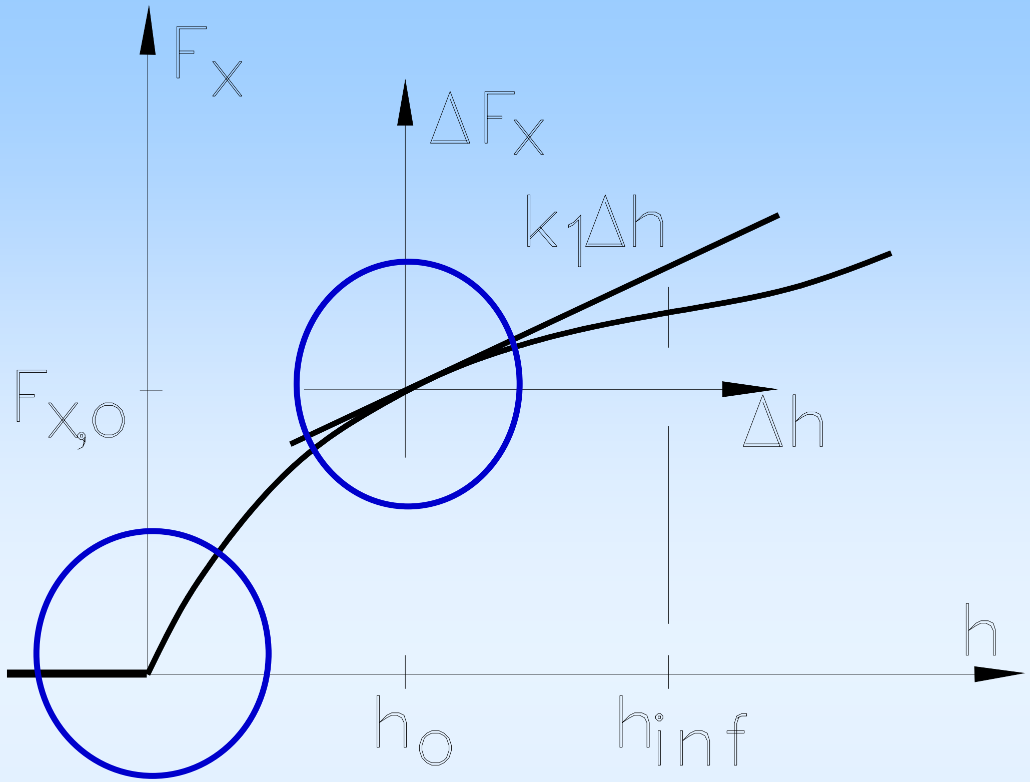


Mathematical model

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}\Delta F_x(\Delta h)$$

Cutting force

¾ rule for nonlinear cutting force



$$F_x(w,h) = c_1 w h^{3/4}$$

$$F_x = F_{x,0} + k_1 \Delta h + k_2 (\Delta h)^2 + k_3 (\Delta h)^3 + \ldots$$

Cutting coefficient

$$k_1(w,h_0) = \left.\frac{\partial F_x(w,h)}{\partial h}\right|_{h_0} = \frac{3}{4} c_1 w h_0^{-1/4}$$

$$k_2 = -\frac{1}{8}\frac{k_1}{h_0}$$

$$k_3 = \frac{5}{96}\frac{k_1}{h_0^2}$$

Linear analysis – stability

$$\ddot{x}(t)+2\xi\omega_n\dot{x}(t)+(\omega_n^2+\frac{k_1}{m})x(t)=\frac{k_1}{m}x(t-\tau)$$

Dimensionless time $\tilde{t}=\omega_n t$

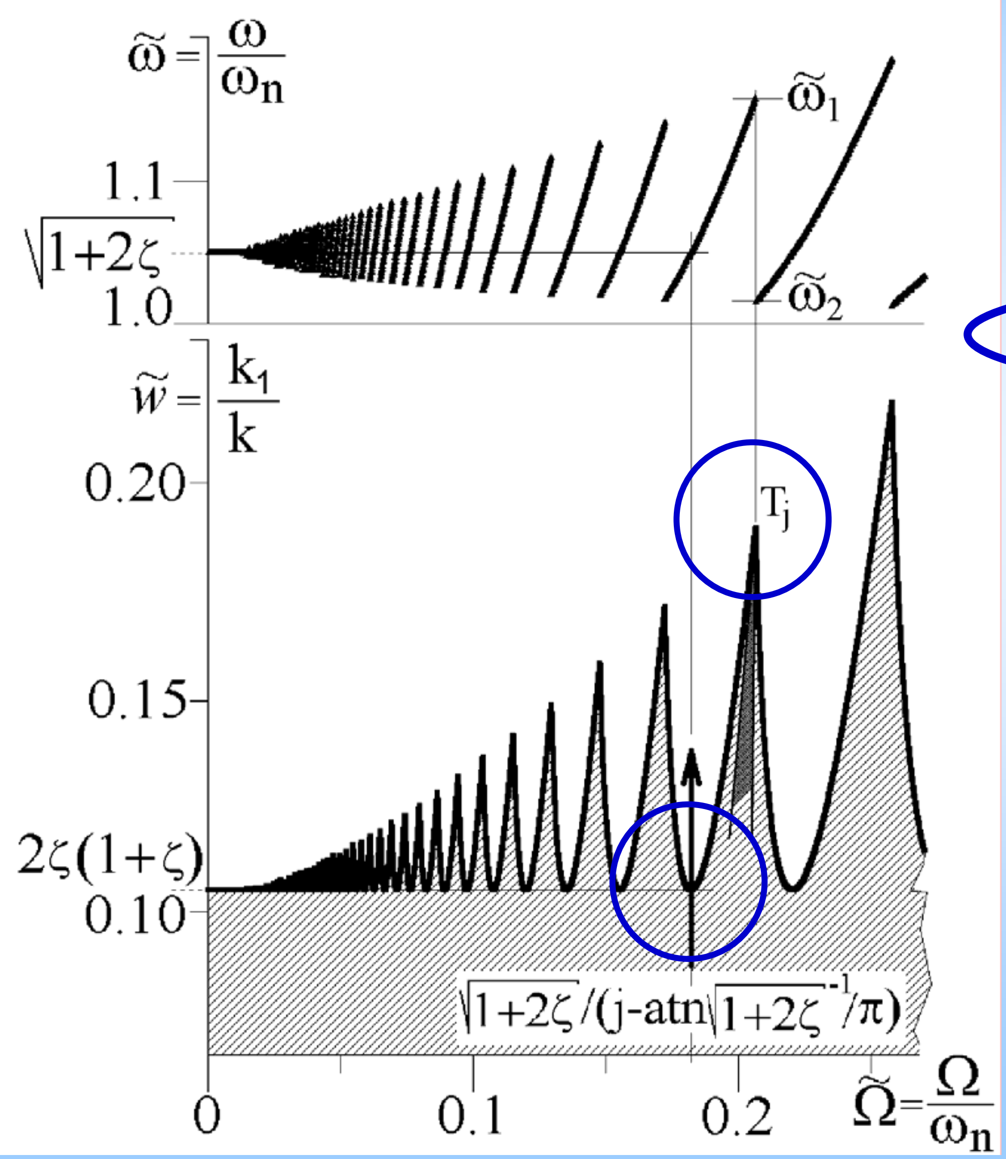
$$x''(\tilde{t})+2\xi x'(\tilde{t})+(1+\tilde{w})x(\tilde{t})=\tilde{w}x(\tilde{t}-\omega_n\tau)$$

Dimensionless chip width $\tilde{w}=\frac{k_1}{m\omega_n^2}=\frac{k_1}{k}$

Dimensionless cutting speed

$$\frac{2\pi}{\tilde{\tau}}=\frac{2\pi}{\omega_n\tau}=\frac{2\pi}{\omega_n\frac{2\pi}{\Omega}}=\frac{\Omega}{\omega_n}$$

Stability and bifurcations of turning



Subcritical Hopf bifurcation: *unstable vibrations* around stable cutting

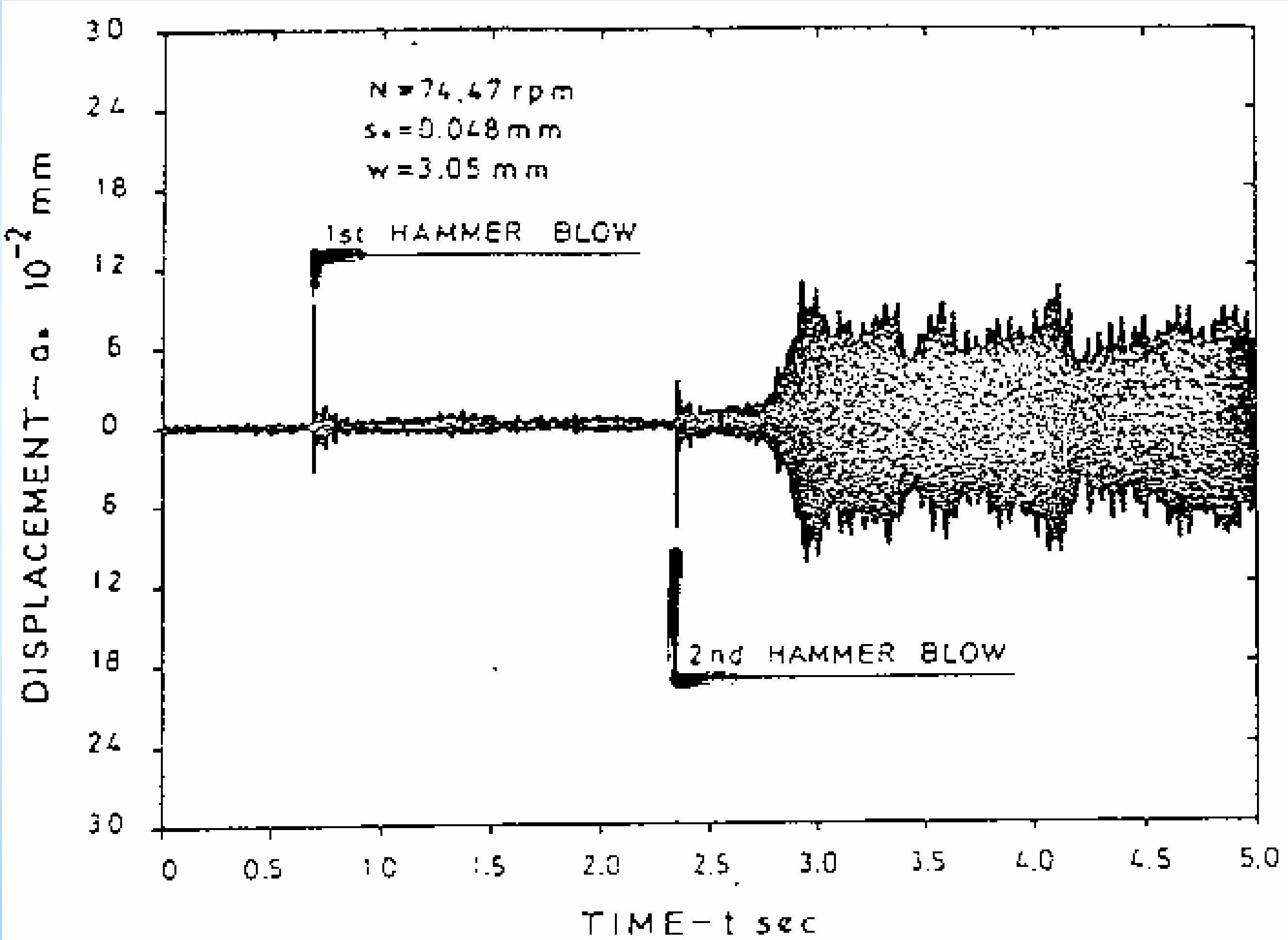
$$\frac{\Omega_j}{\omega_n} = \frac{\sqrt{1+2\xi}}{j - \frac{1}{\pi}\,\text{atn}\,\frac{1}{\sqrt{1+2\xi}}}$$

$$\tilde{w}_{cr} = 2\xi(1+\xi)$$

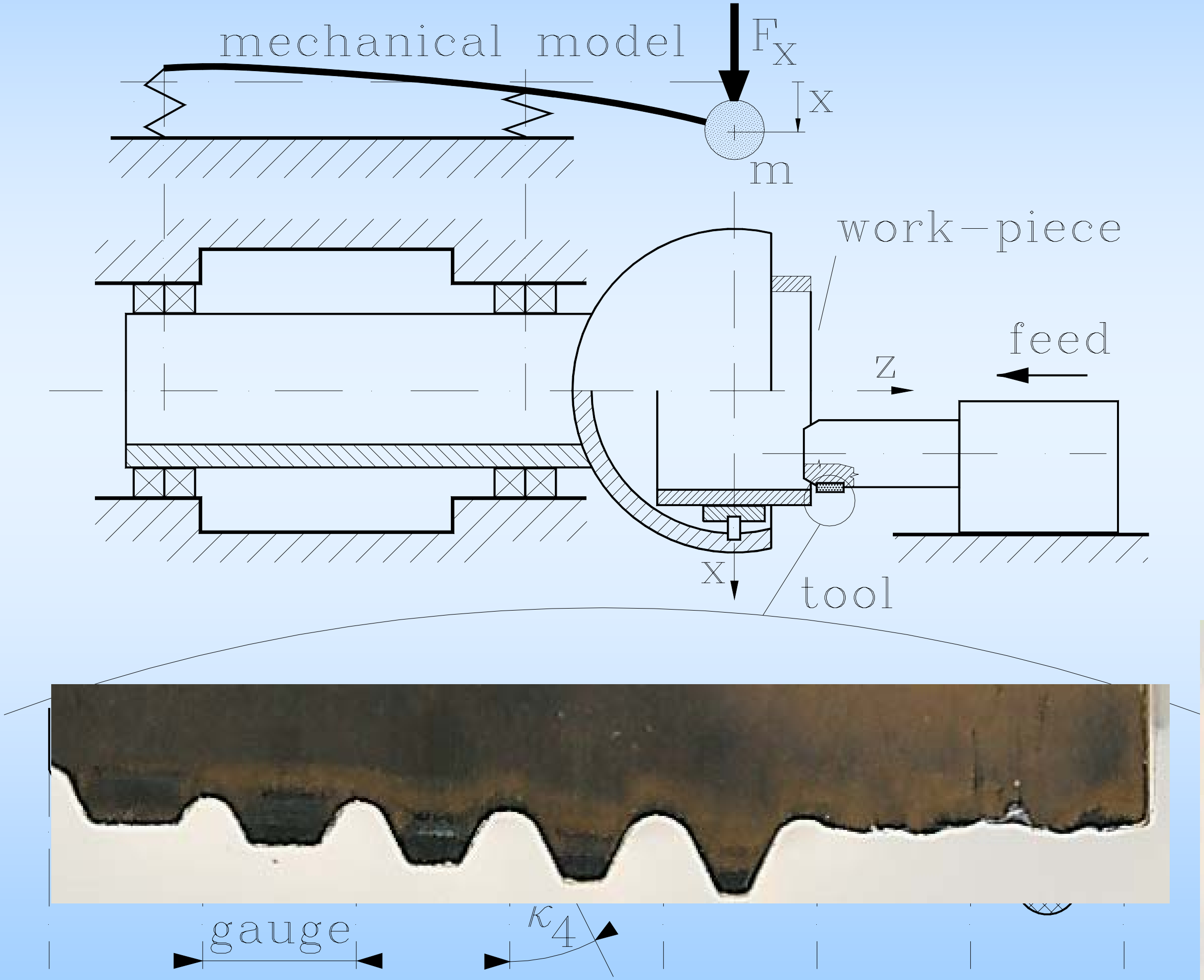
$$\omega_{cr} = \omega_n\sqrt{1+2\xi}$$

The unstable periodic motion

Shi, Tobias (1984) –
impact experiment



Case study – thread cutting



$m = 346$ [kg]

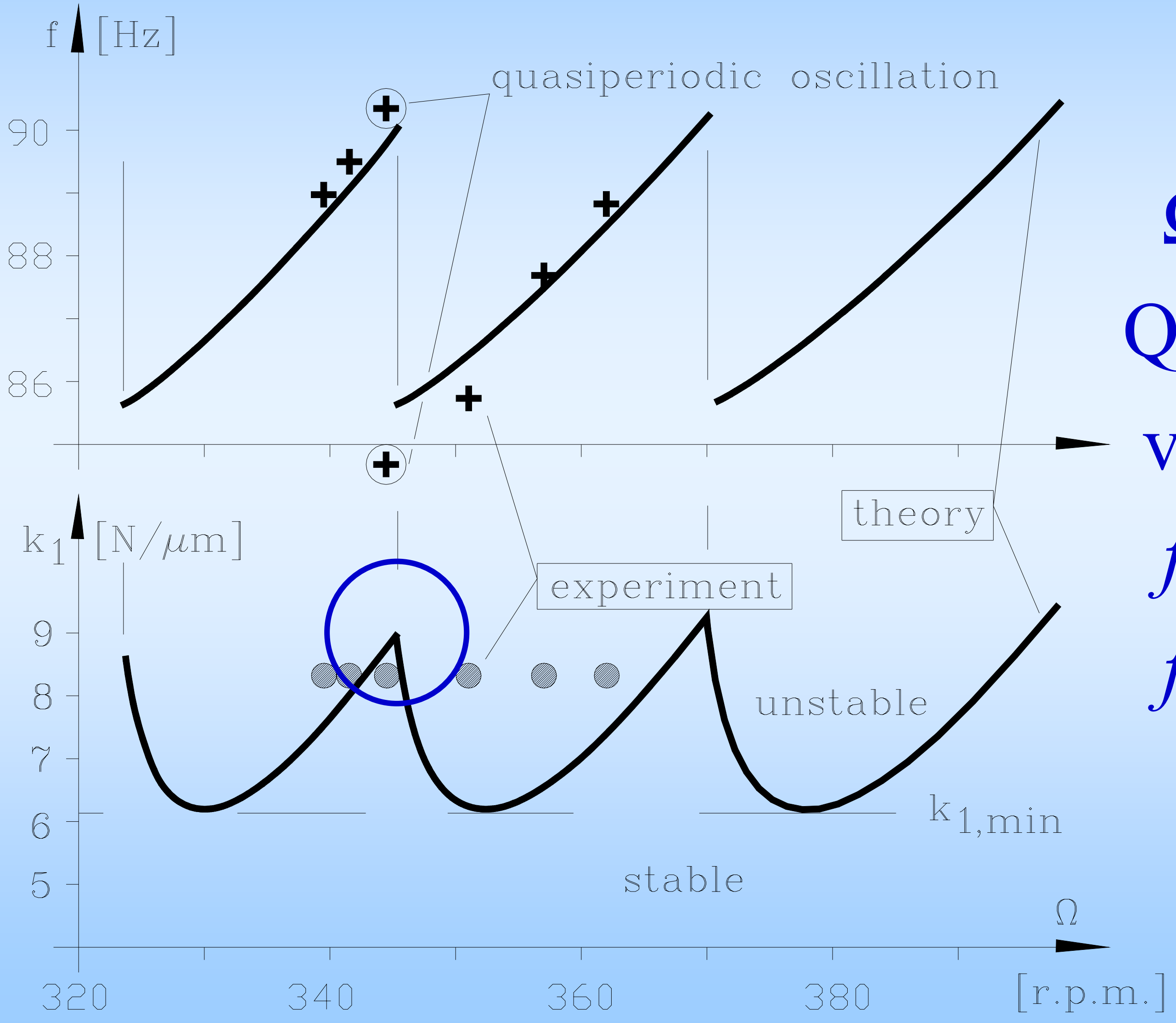
$k = 97$ [N/μm]

$f_n = 84.1$ [Hz]

$\xi = 0.025$

gge=3.175[mm]

Stability of thread cutting – theory&exp.



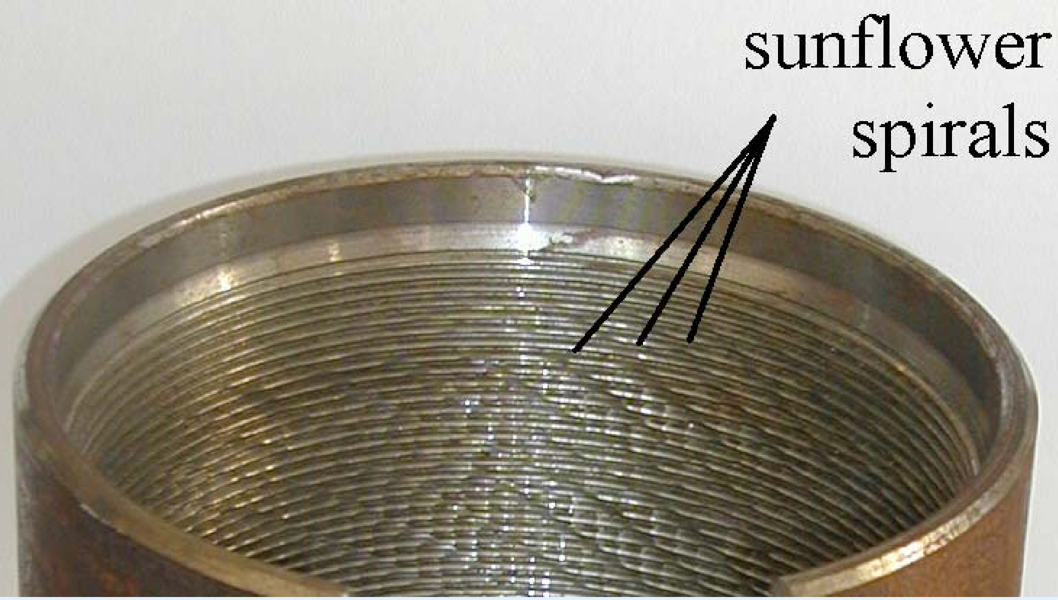
Ω=344 f/p

Quasi-periodic vibrations:

f_1=84.5 [Hz]

f_2=90.8 [Hz]

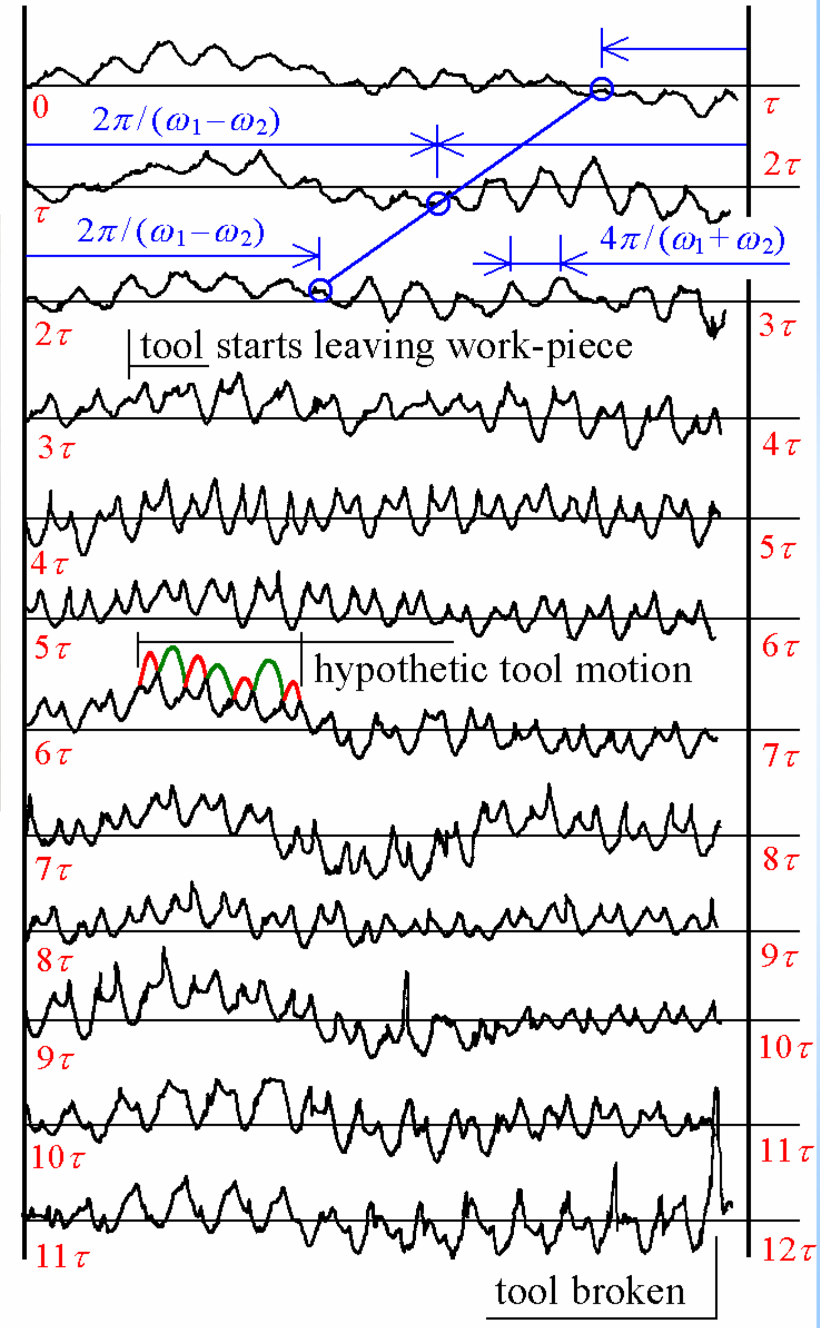
Machined surface



D=176 [mm], τ =0.175 [s]

$$\frac{f_1+f_2}{2}=\frac{15.3}{\tau}=88.0[\mathrm{Hz}]$$

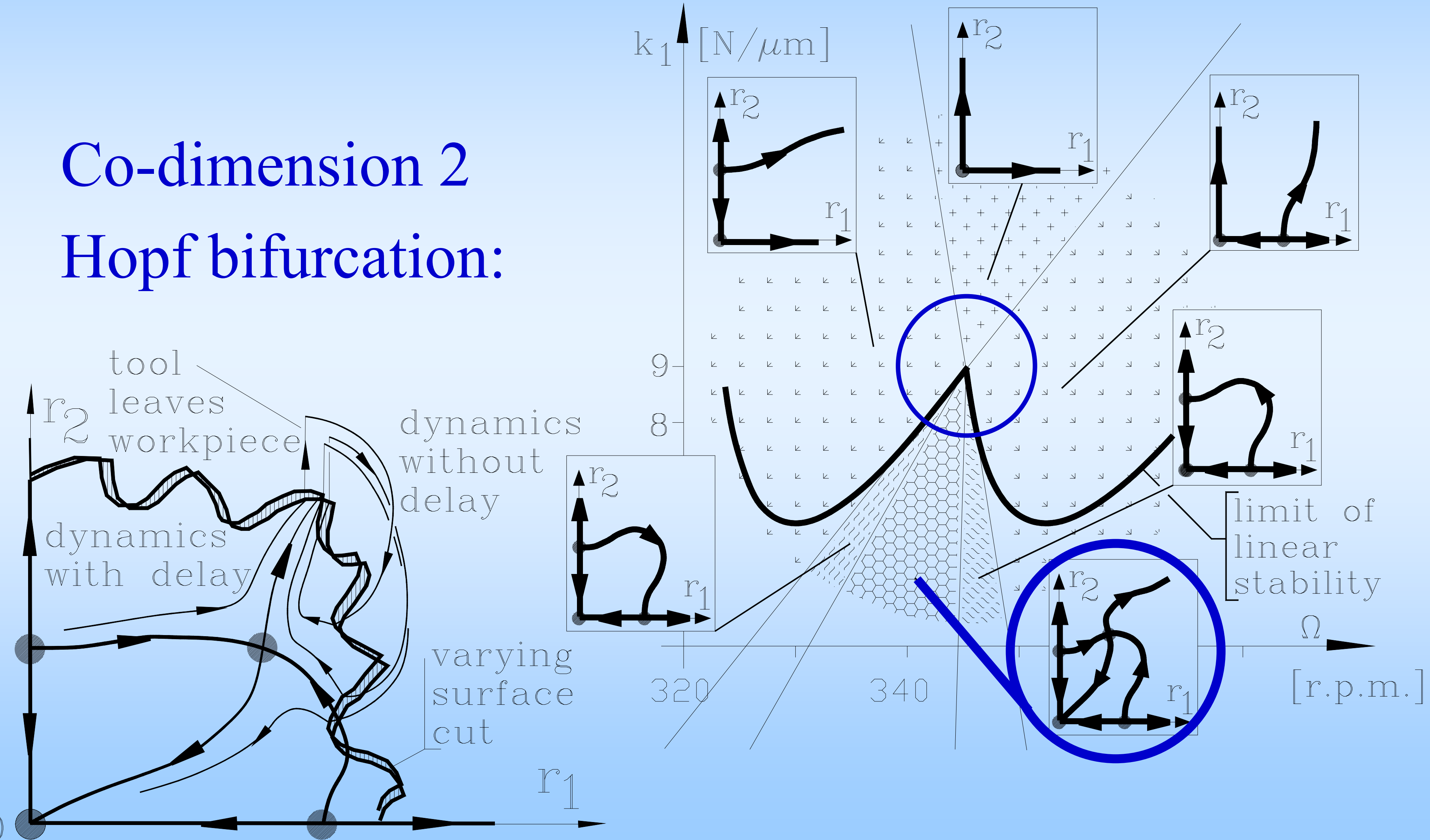
$$\frac{f_1-f_2}{2}=\frac{15.3}{(2\times 12.5)\tau}=3.5[\mathrm{Hz}]$$



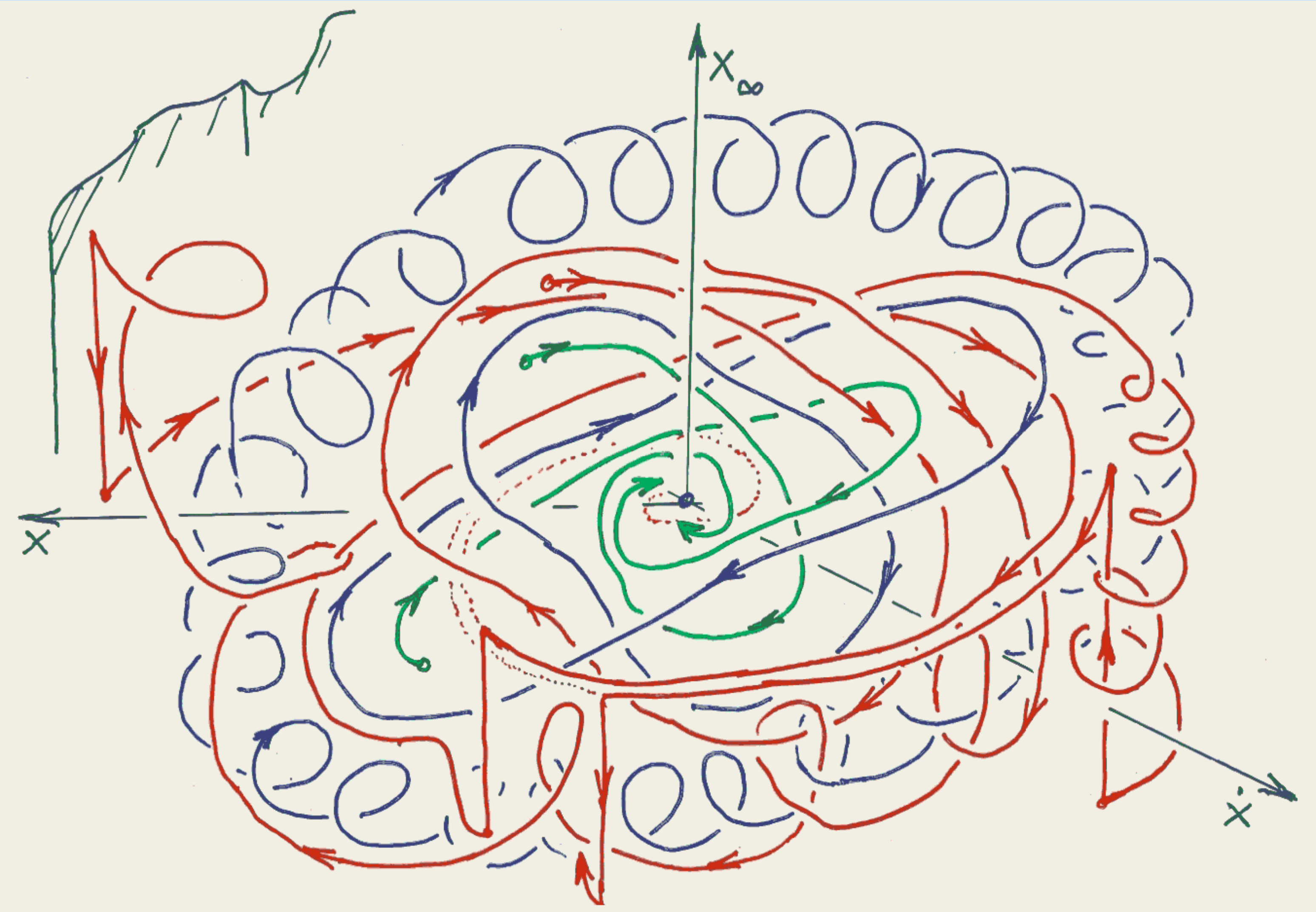
Unstable quasi-periodic vibration

Co-dimension 2

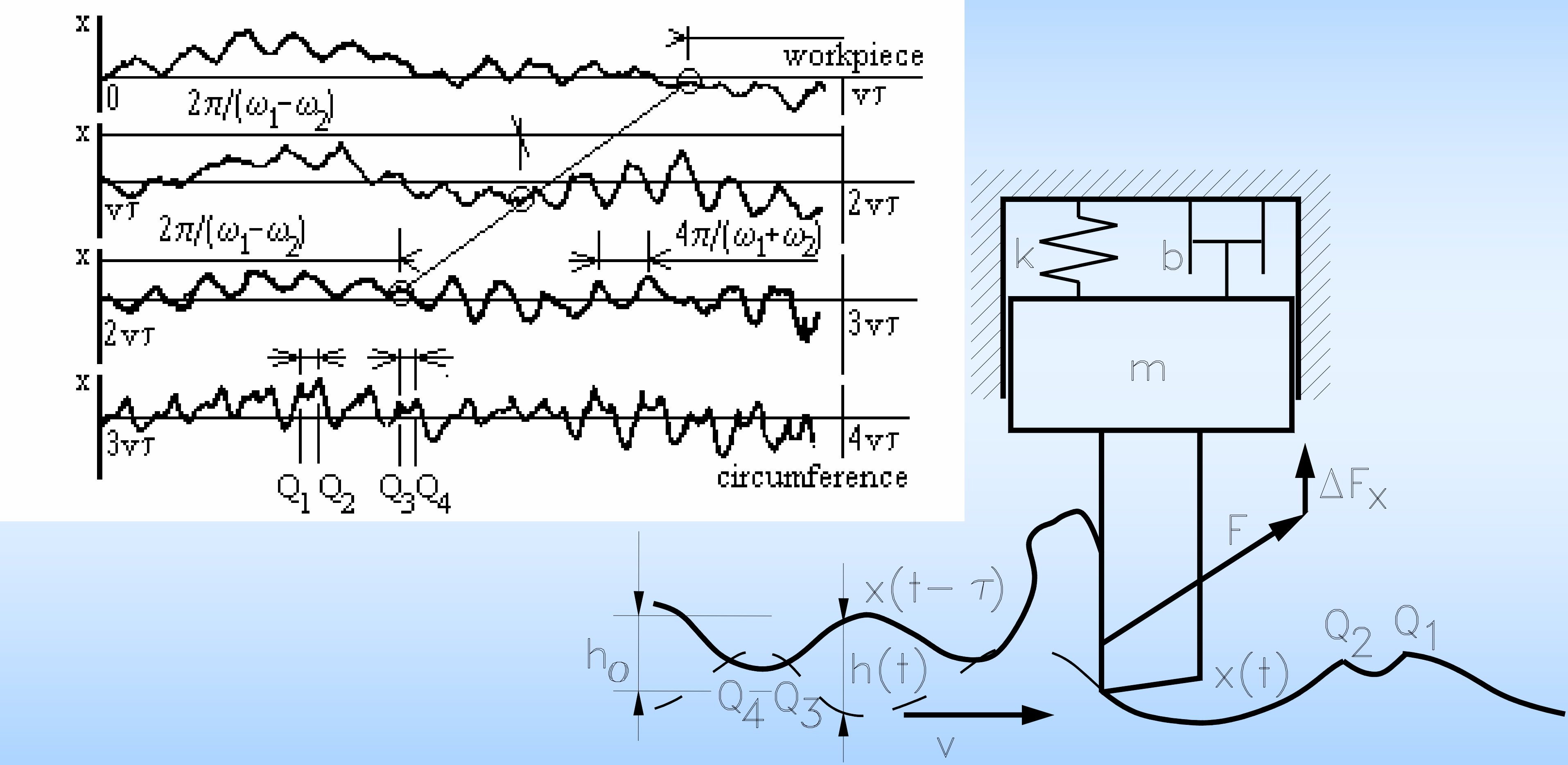
Hopf bifurcation:

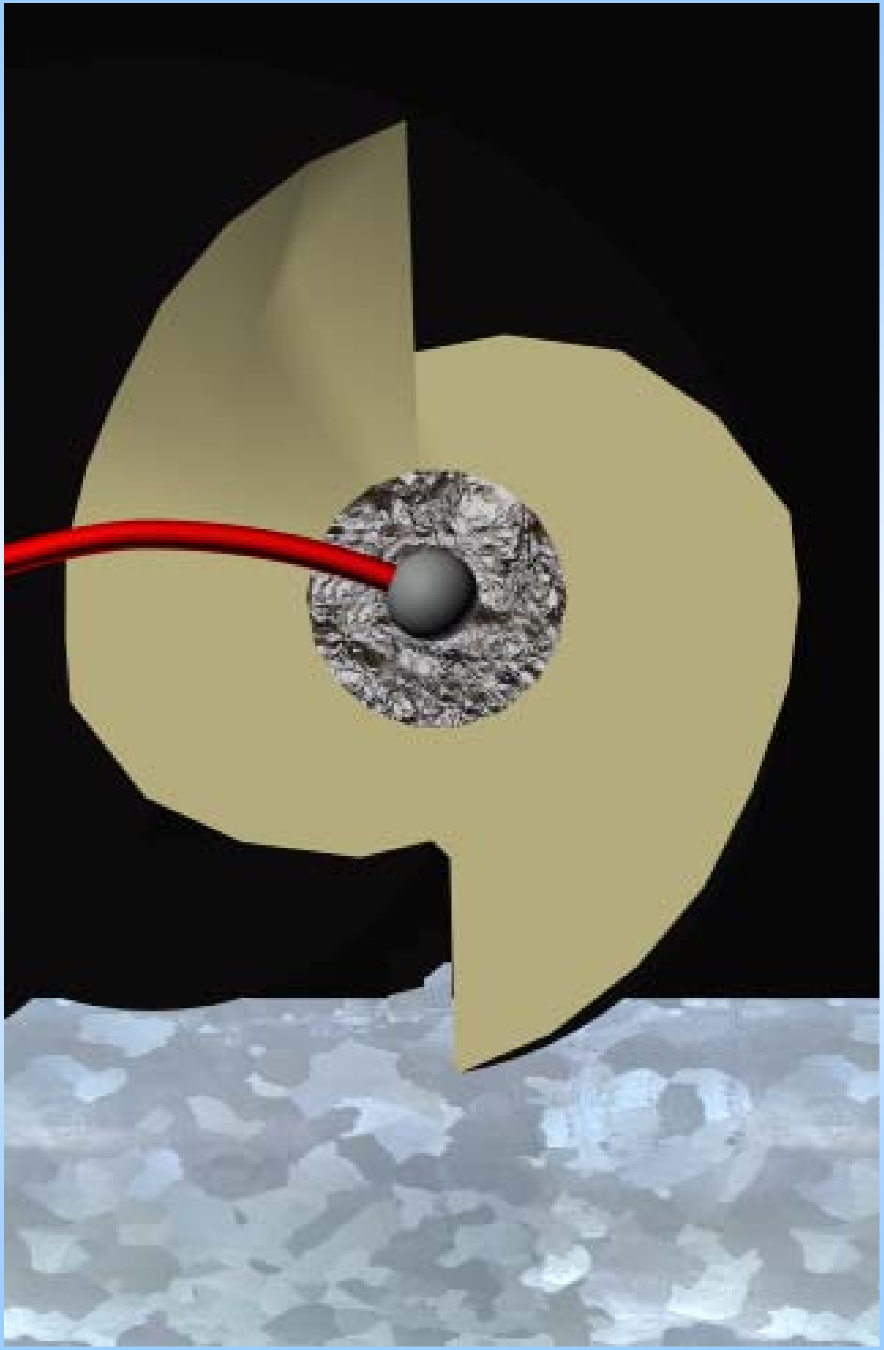


Phase space structure



Self-interrupted cutting





High-speed milling

Parametrically interrupted cutting

Low number of edges

Low immersion

Highly interrupted

Highly interrupted cutting

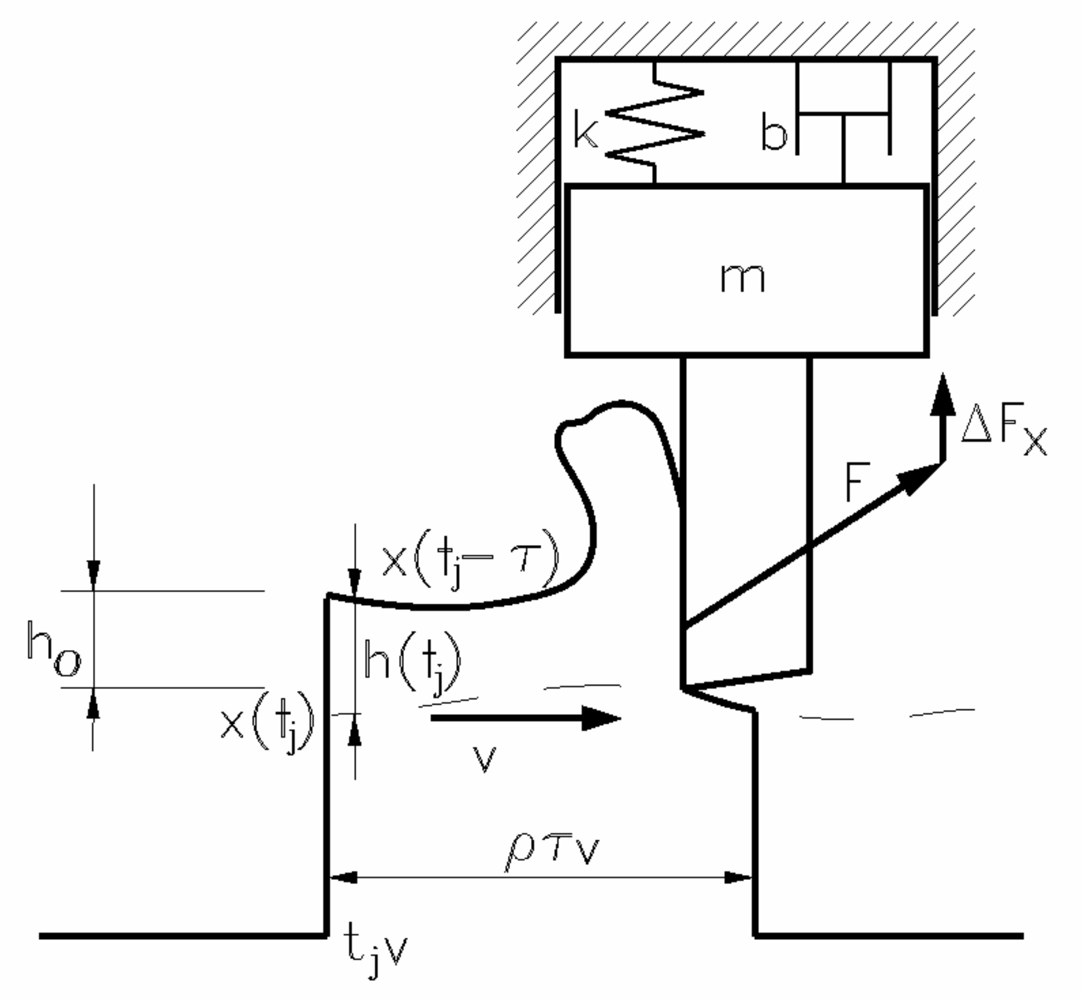
Two dynamics:

- free-flight

  $t \in [t_j - \tau \,, t_j - \rho\tau)$

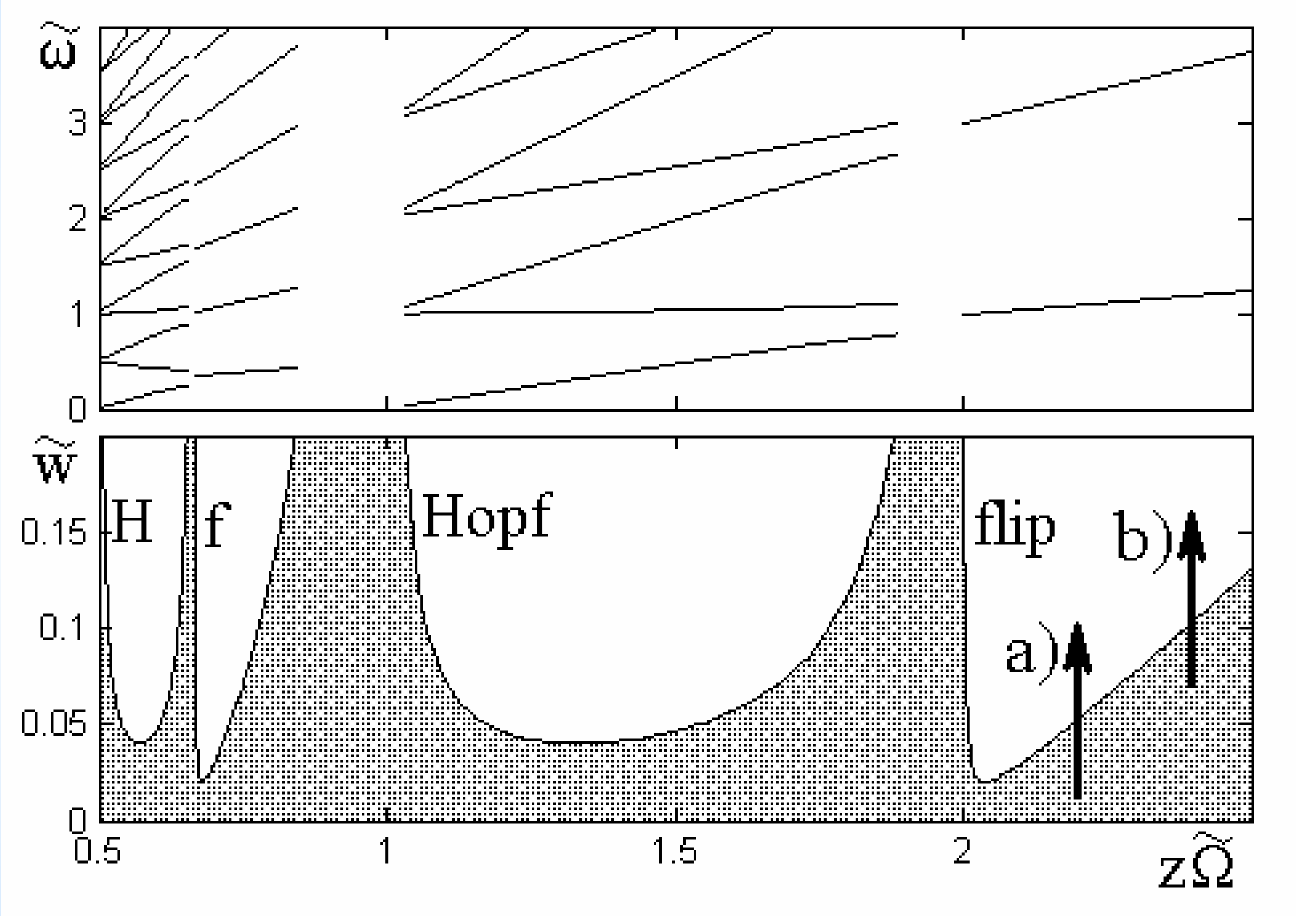
- cutting with regenerative effect – like an impact

  $t \in [t_j - \rho\tau \,, t_j \,)$



$$\begin{bmatrix} x_j \\ v_j \end{bmatrix} = \textcircled{\mathbf{A}} \begin{bmatrix} x_{j-1} \\ v_{j-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \sum\limits_{h+k=2,3;\ h,k\ge 0} b_{hk}\, x_{j-1}^{h}\, v_{j-1}^{k} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\rho\tau}{m} F_0 \end{bmatrix}$$

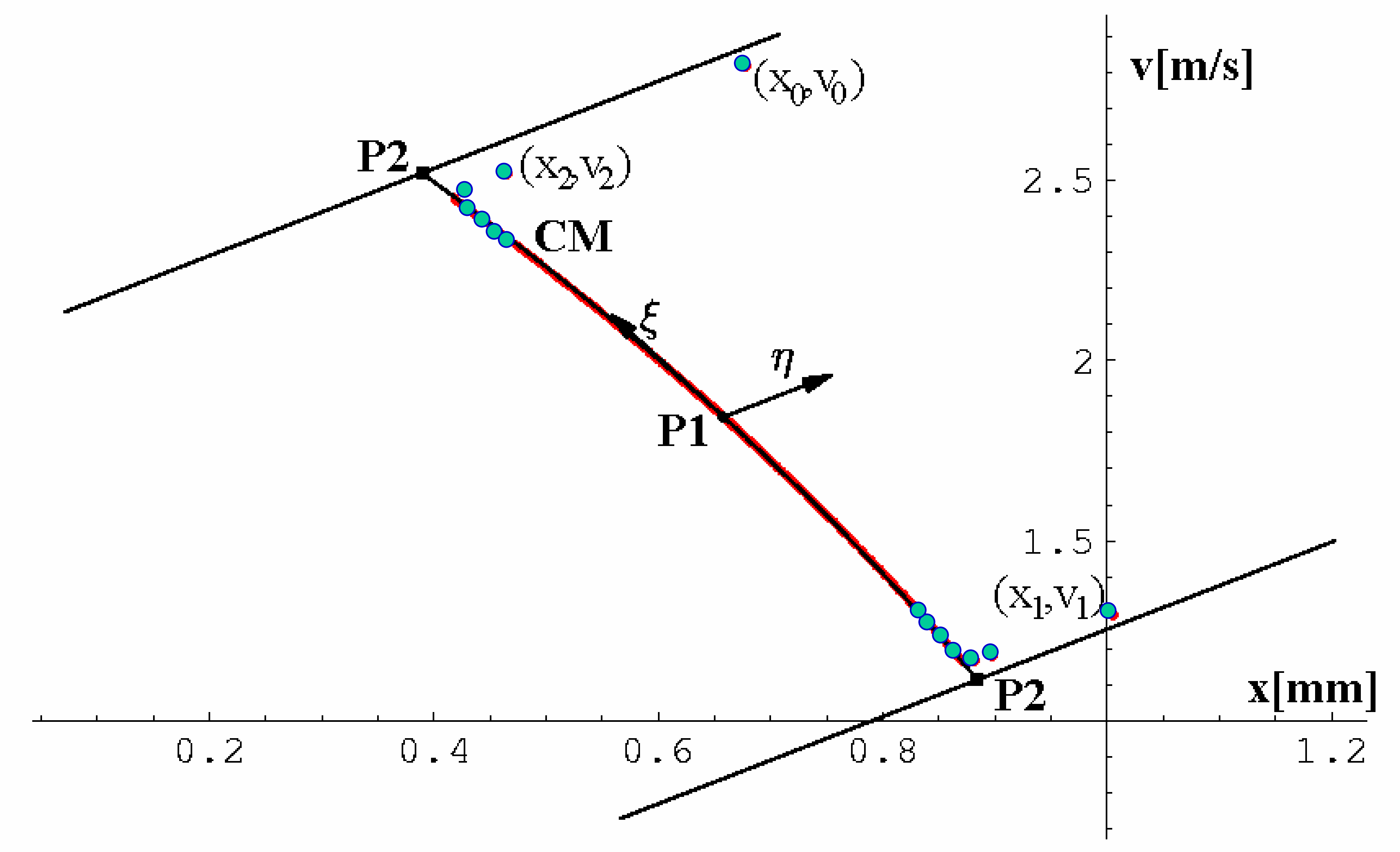
Stability chart of H-S milling



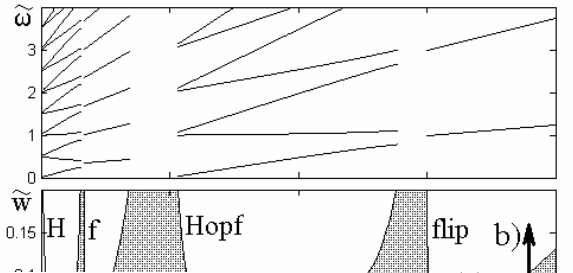
Sense of the period doubling (or flip) bifurcation?

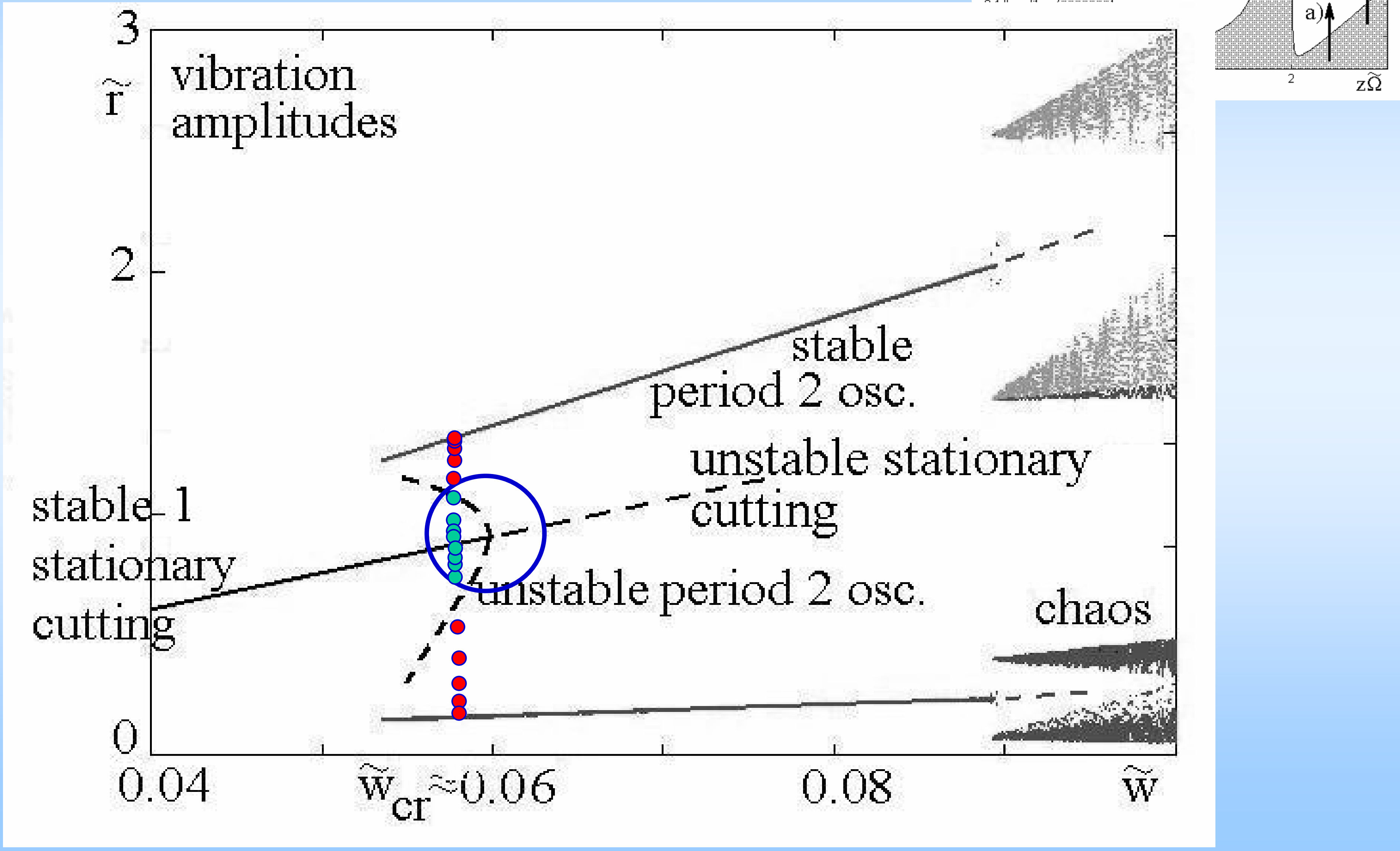
(Davies, Burns, Pratt, 2000)

Subcritical flip bifurcation

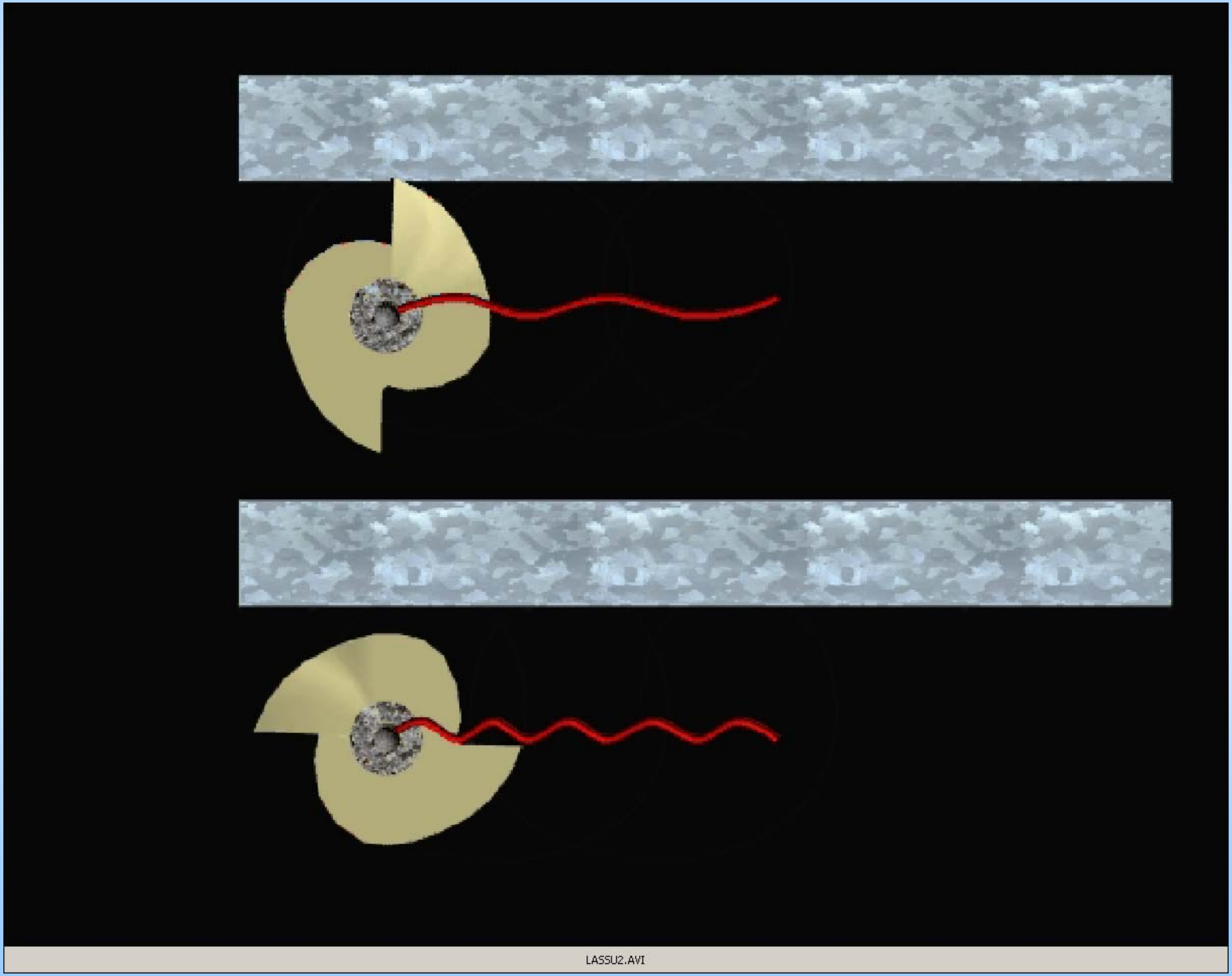


Bifurcation diagram – chaos

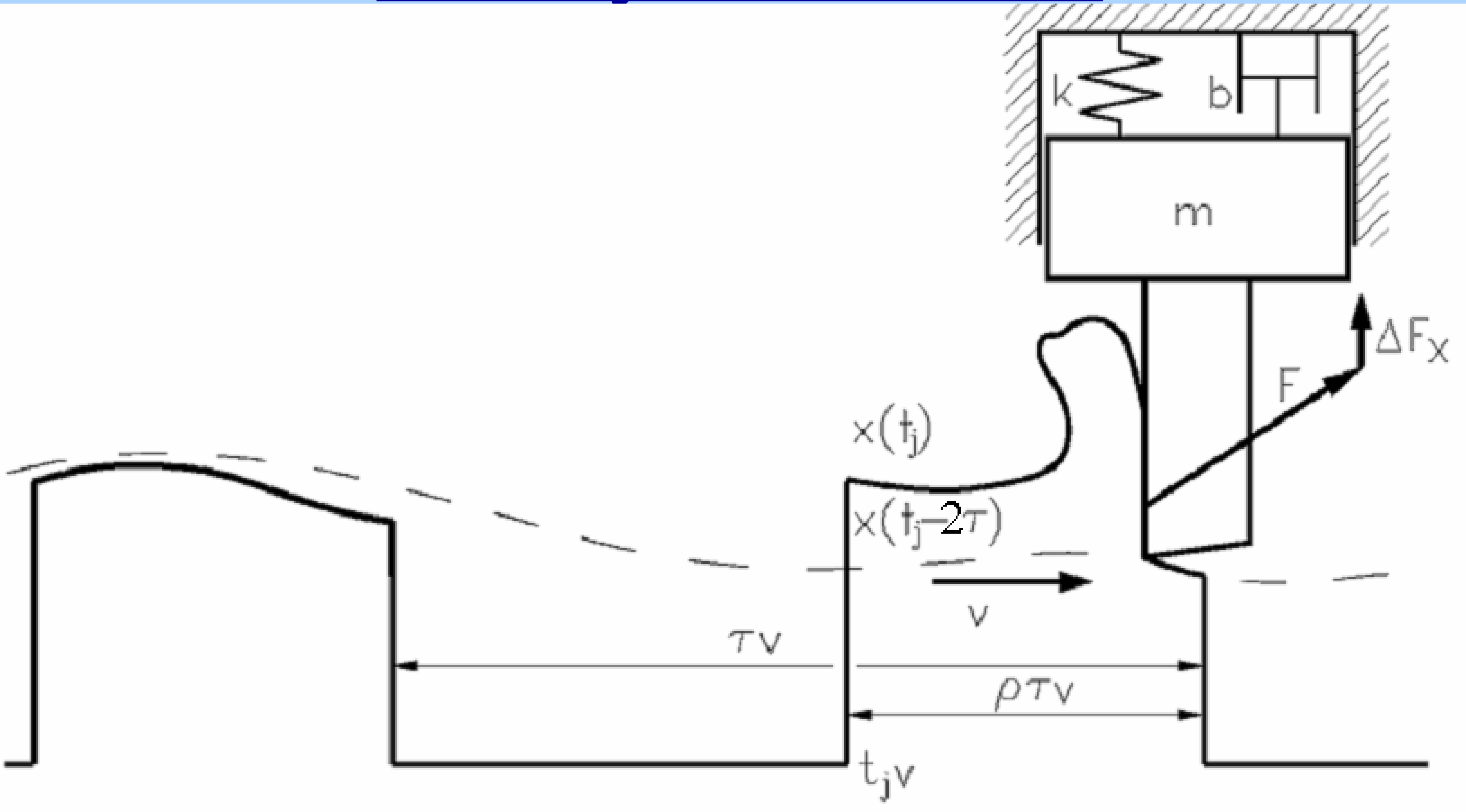




Animation of stable period doubling

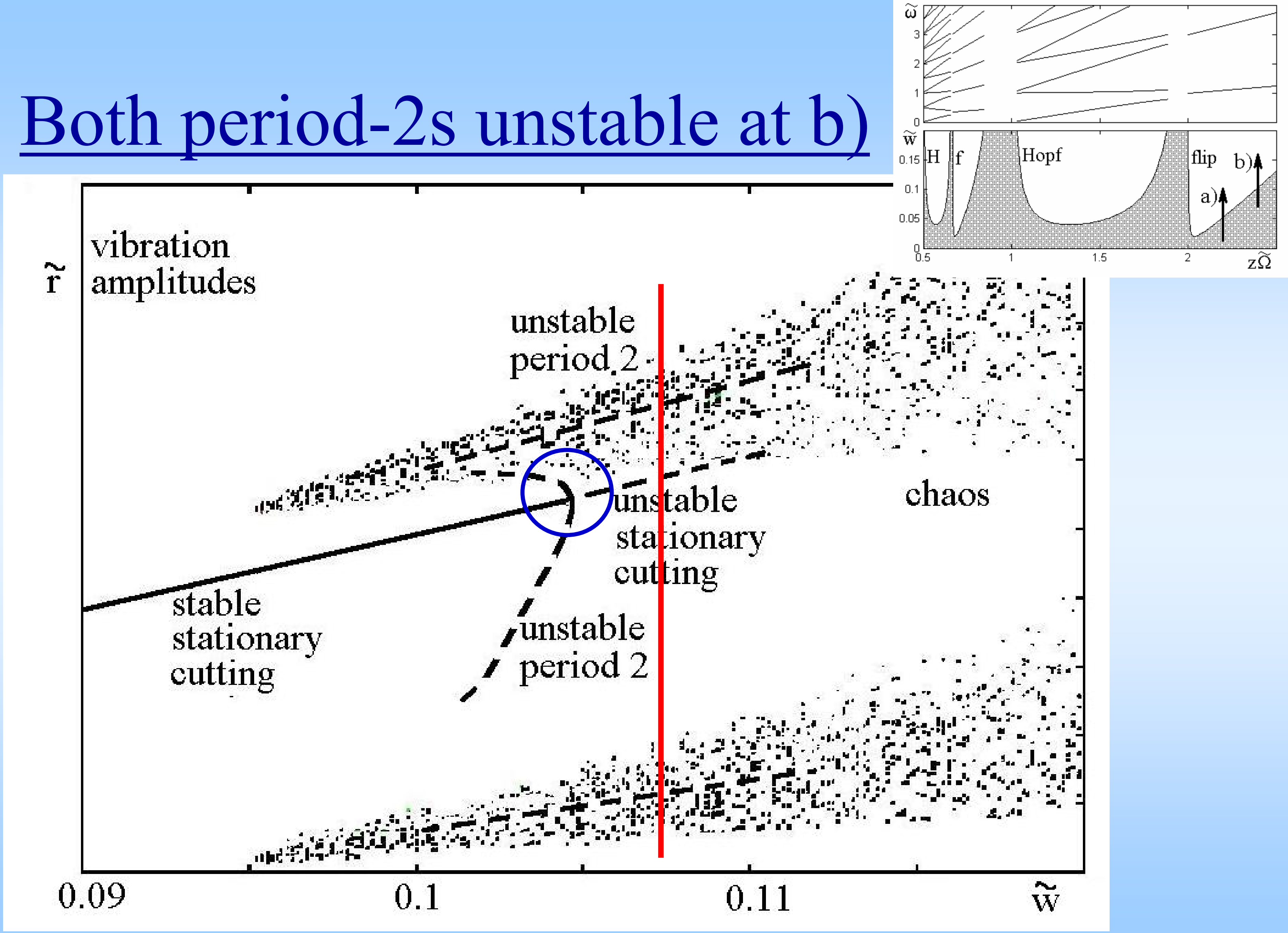


The fly-over effect

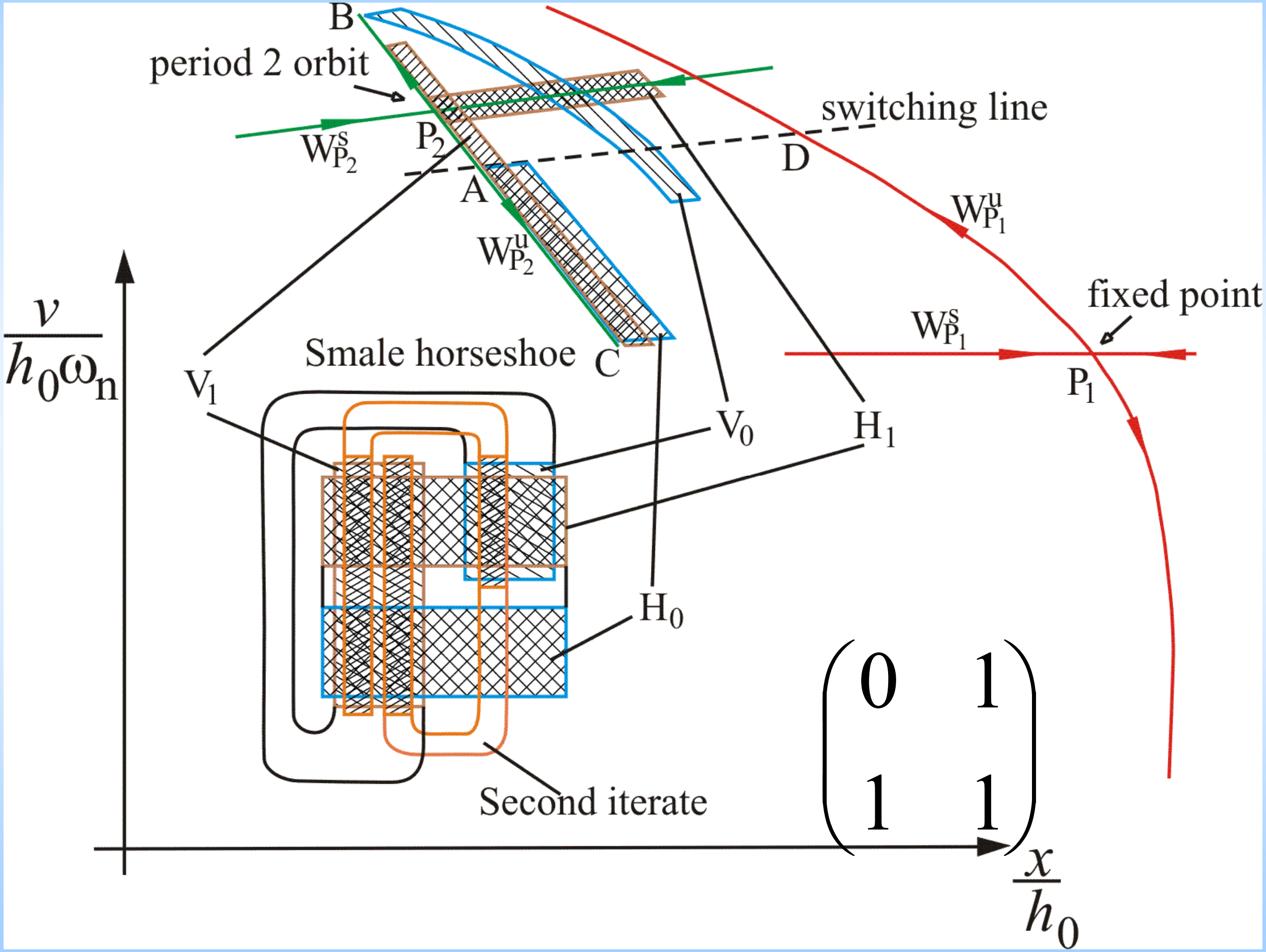


$$\begin{bmatrix} x_j \\ v_j \end{bmatrix} = \mathbf{A}\begin{bmatrix} x_{j-1} \\ v_{j-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \sum\limits_{h+k=2,3;\ h,k\ge 0} b_{hk}\, x_{j-1}^{h}\, v_{j-1}^{k} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{\rho\tau}{m} F_0 \end{bmatrix}$$

Both period-2s unstable at b)



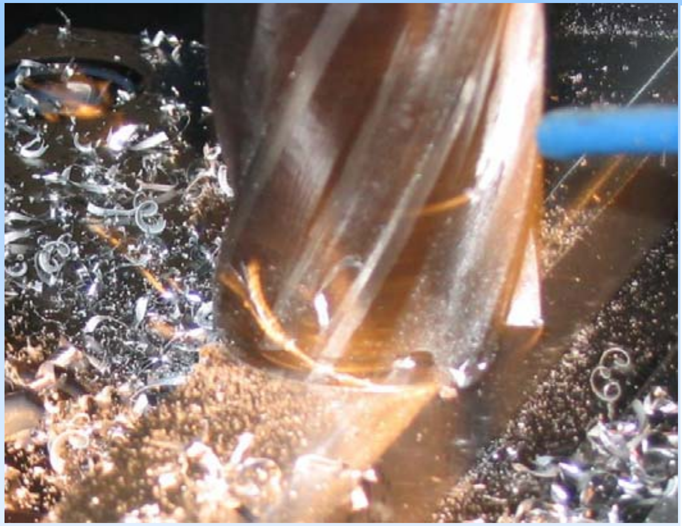
Structure of chaos – transition matrix

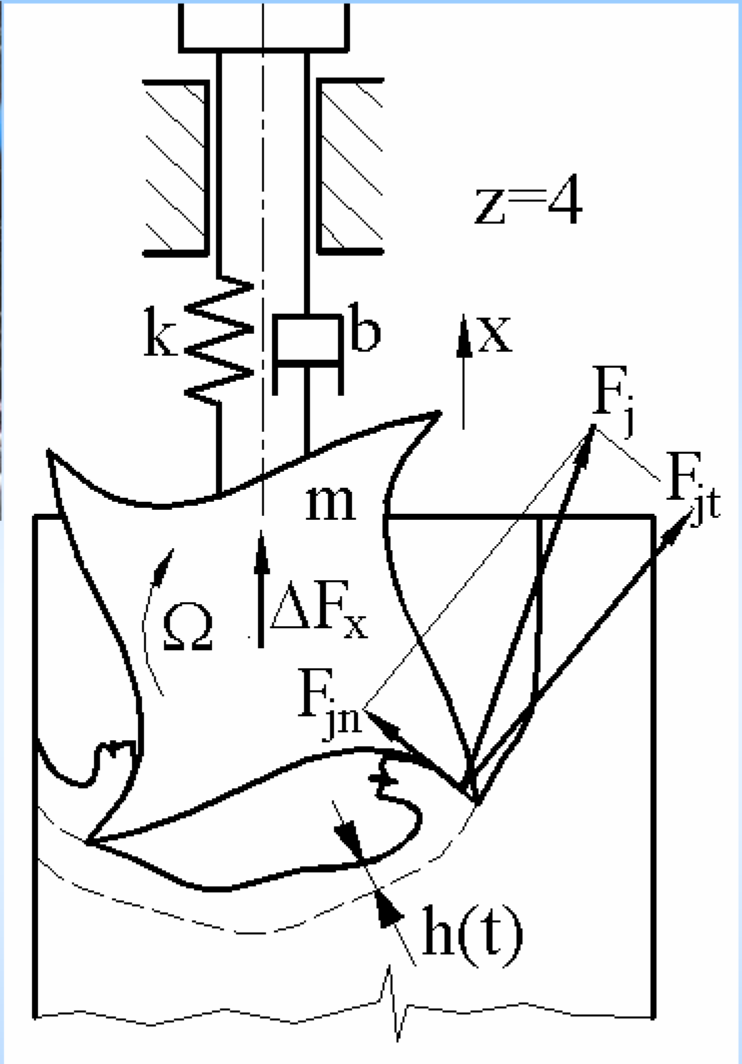


$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Szalai, ~, Hogan (2004)

Milling

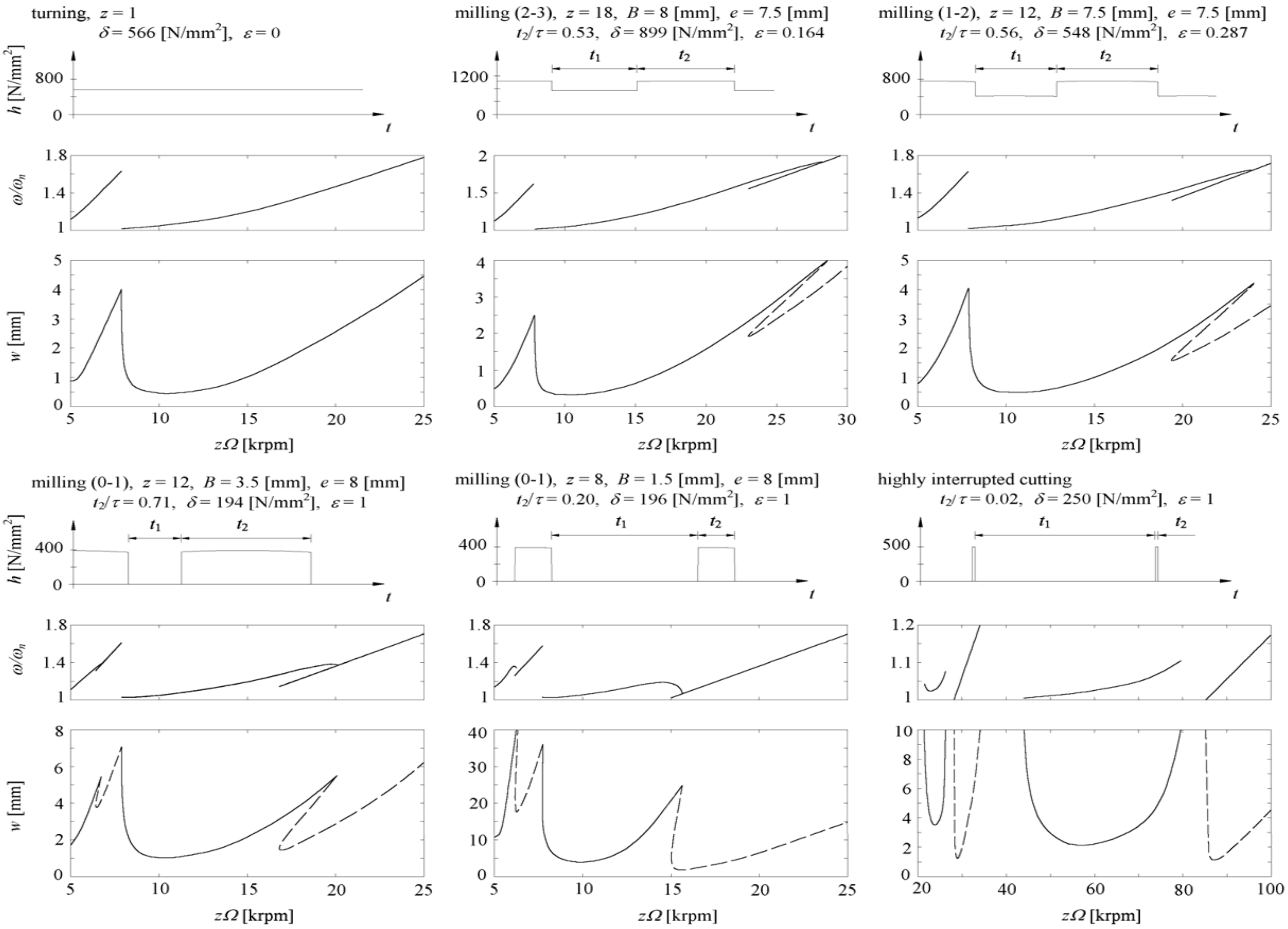




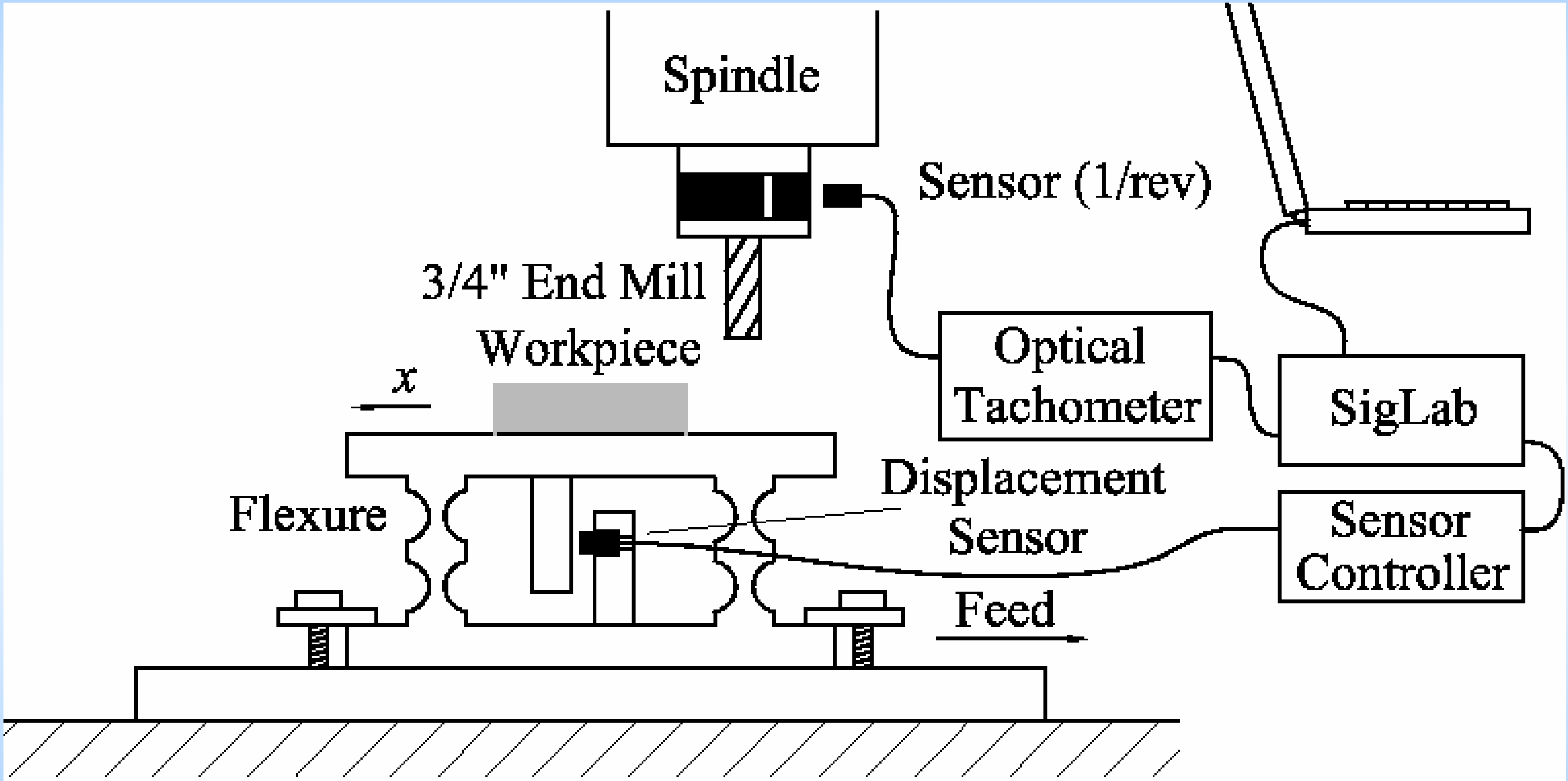
Mechanical model:

- number of cutting edges in contact varies periodically with period equal to the delay

$$\ddot{x}(t)+2\xi\omega_n\dot{x}(t)+(\omega_n^2+\frac{k_1(t)}{m})x(t)=\frac{k_1(t)}{m}x(t-\tau)$$



Milling experiments

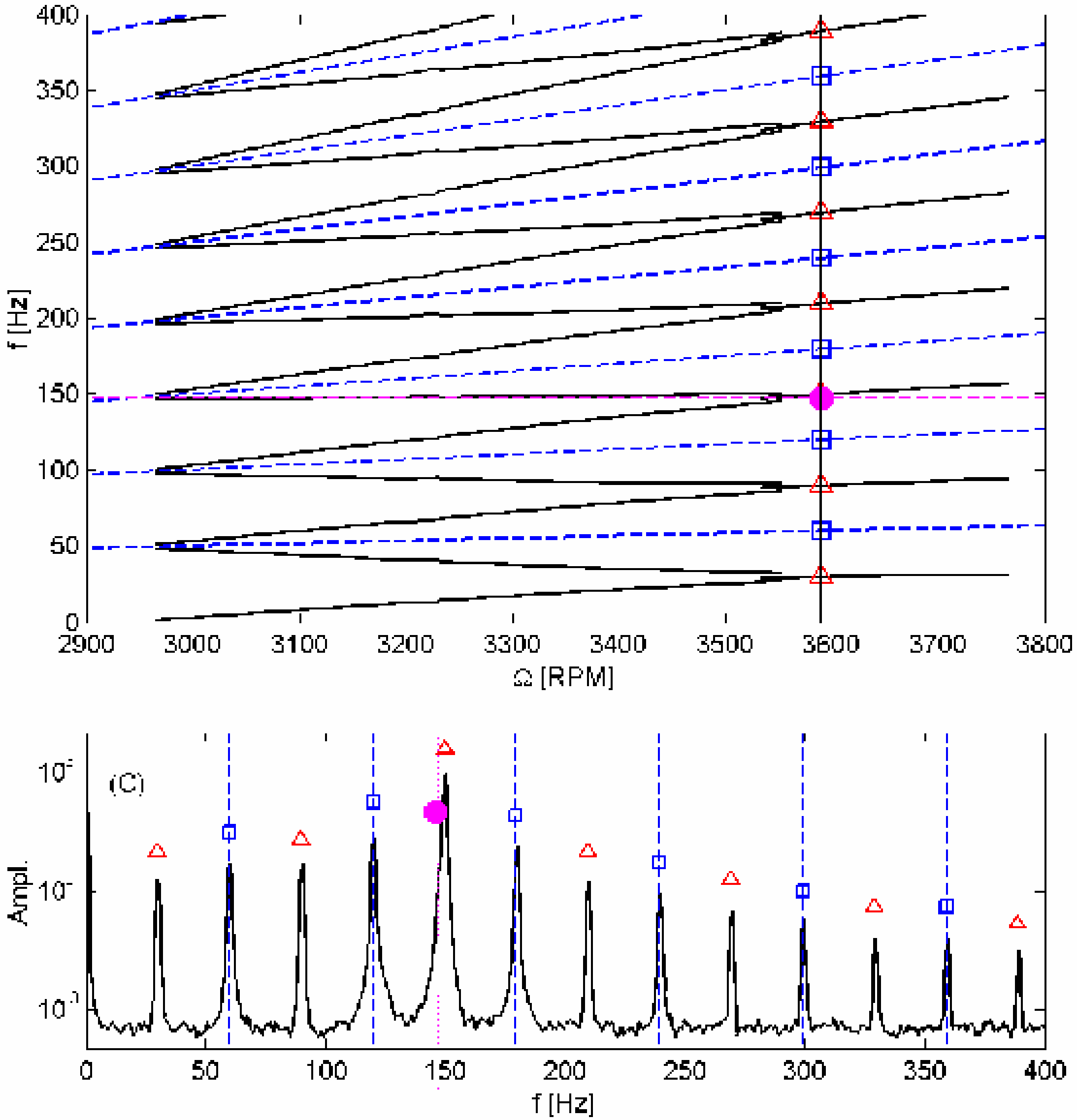


time spent cutting to not-cutting: $\rho = 0.1$

High-speed milling

Theory & experiments:

stability chart and vibration frequencies (Insperger, Mann,~, Bayly, 2003)

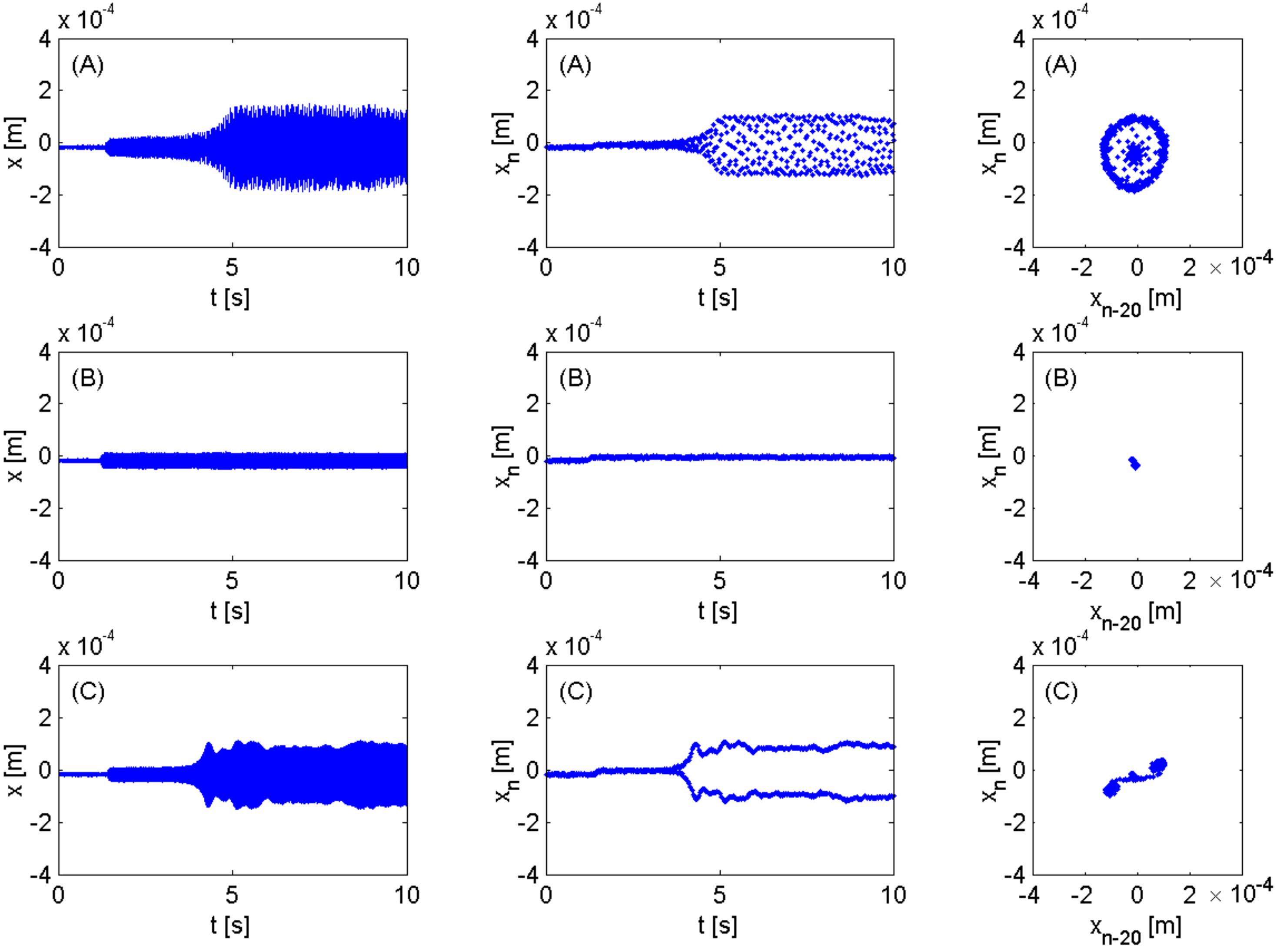


Measured and processed signals

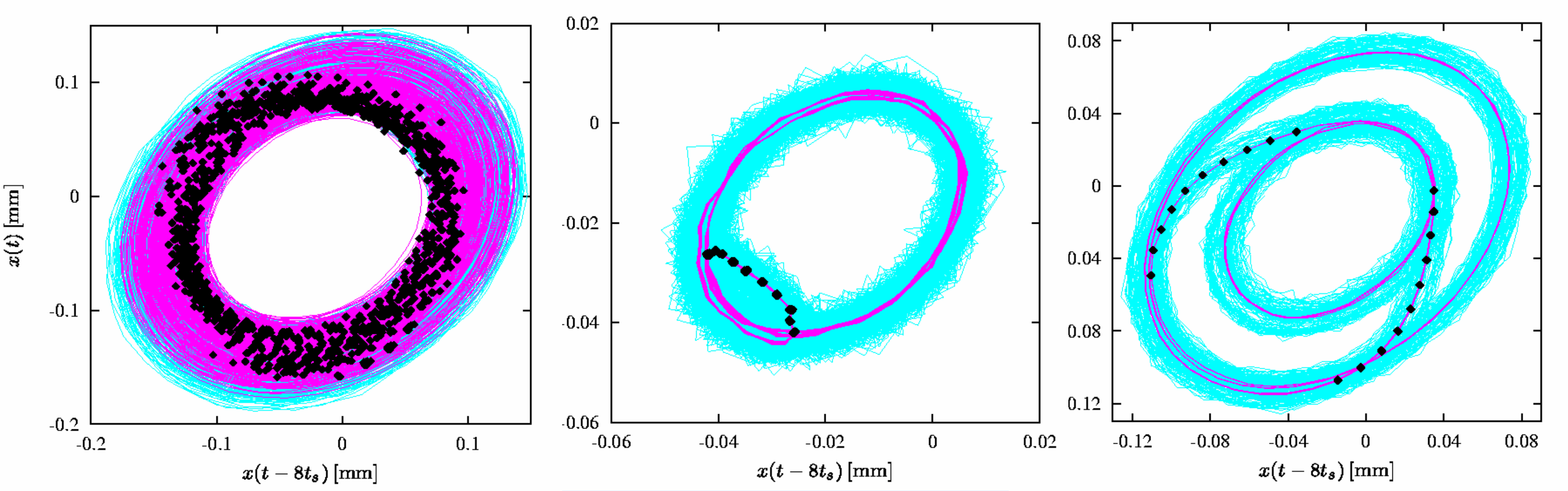
A

B

C



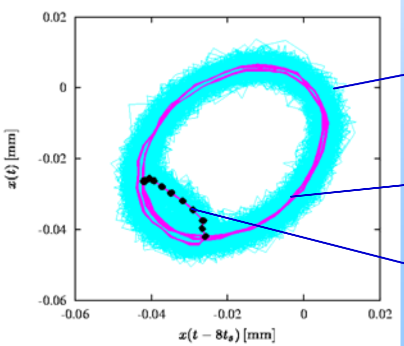
Phase space reconstruction



A – secondary Hopf

B – stable cutting (tooth pass exc.)

C – period-2 osc. (no fly-over!!!)

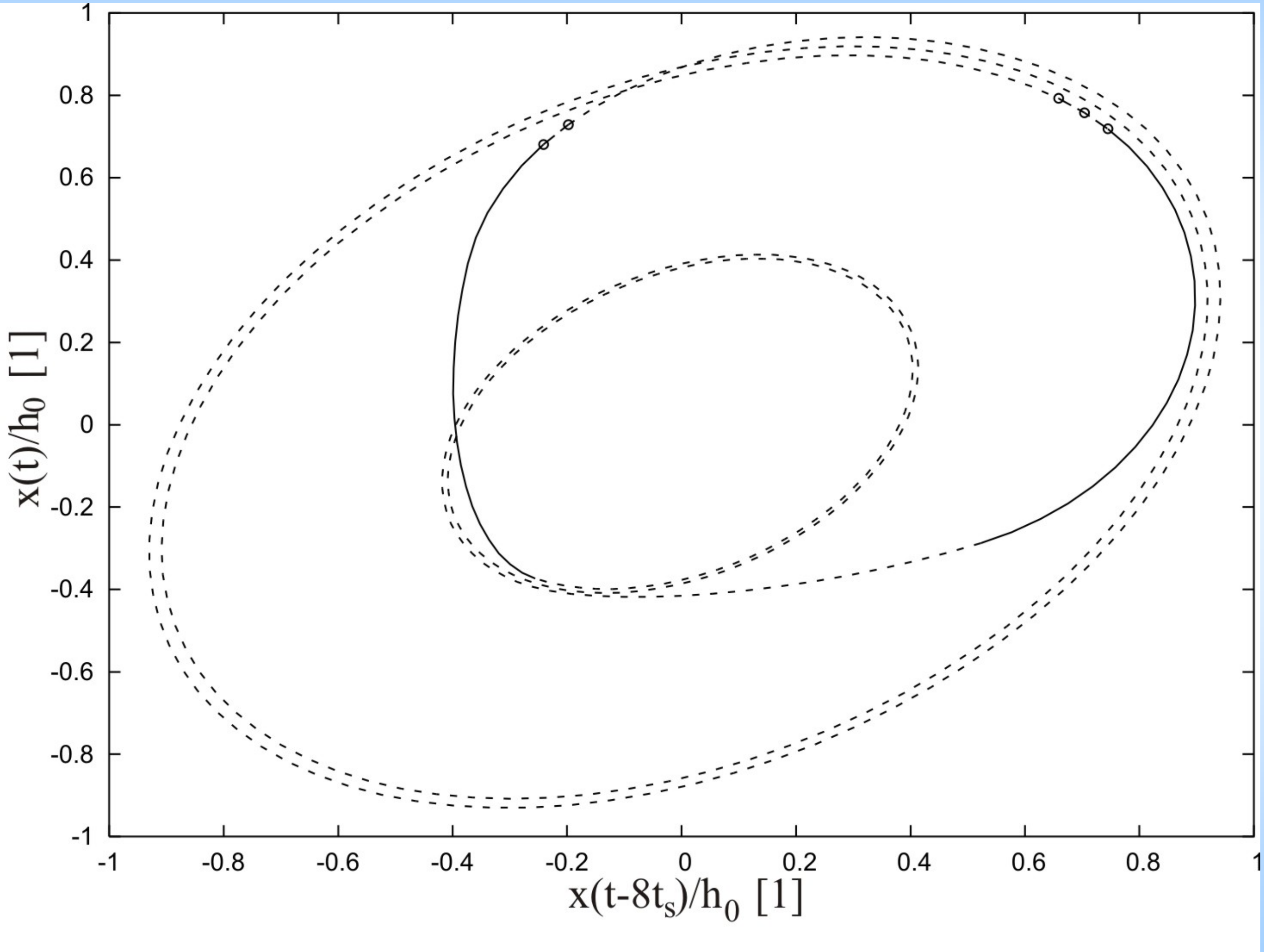


noisy trajectory from measurement

noise-free reconstructed trajectory

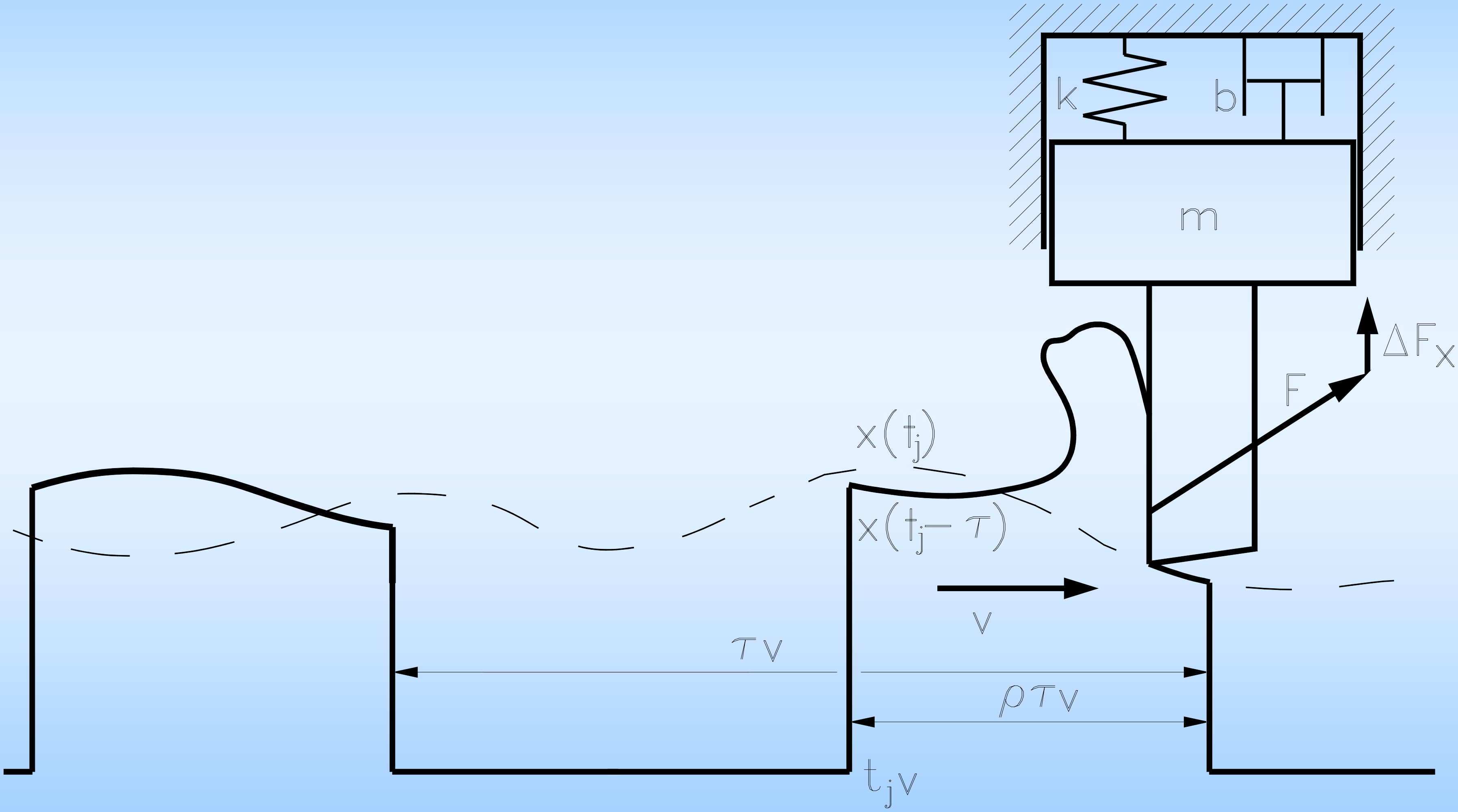
cutting contact(Gradisek,Kalveram)

Collocation method for period-2 motion

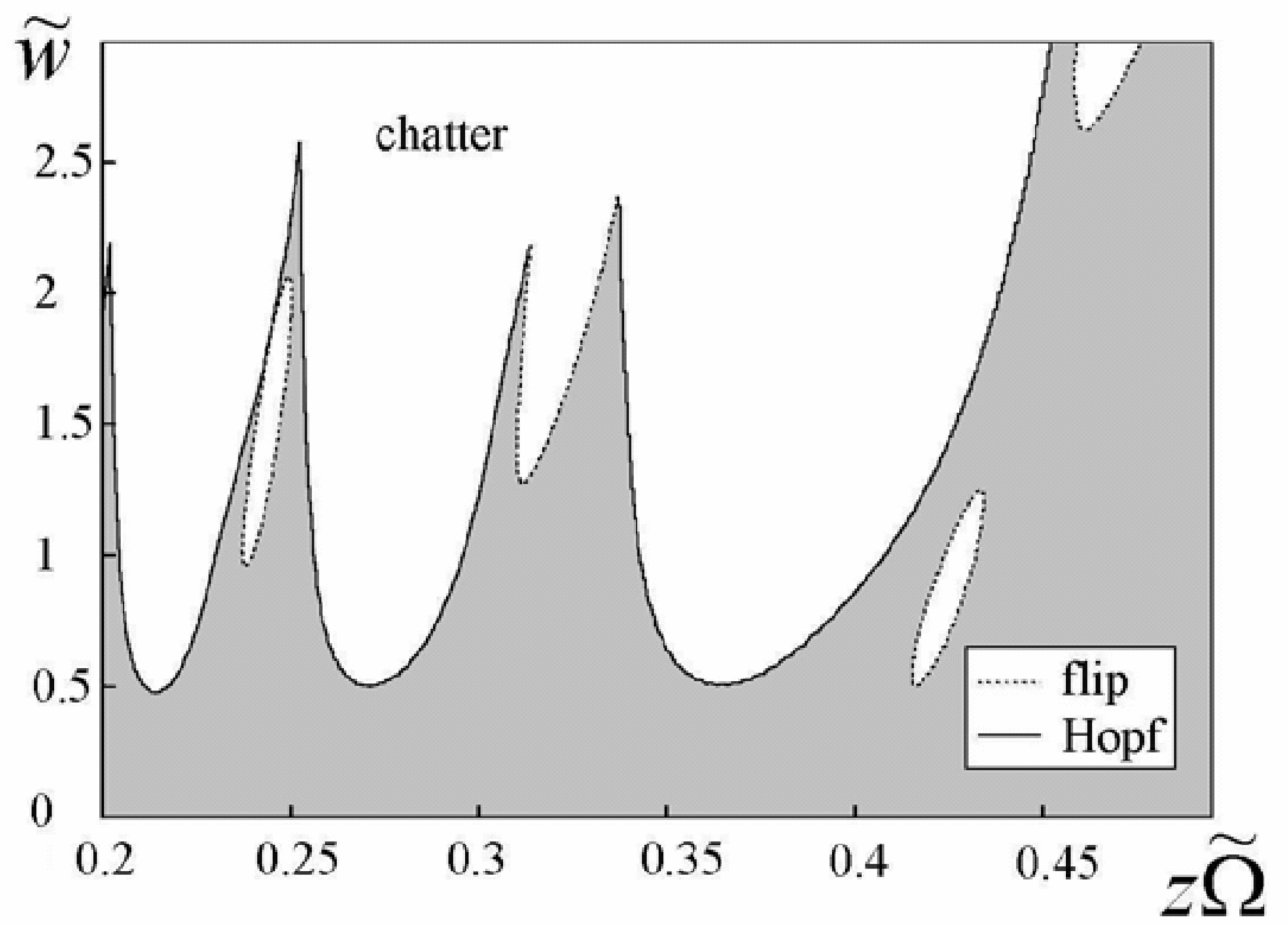


(Engelborghs,Luzyania,Hout,Roose 2000, Szalai, ~)

The stable period-2 motion

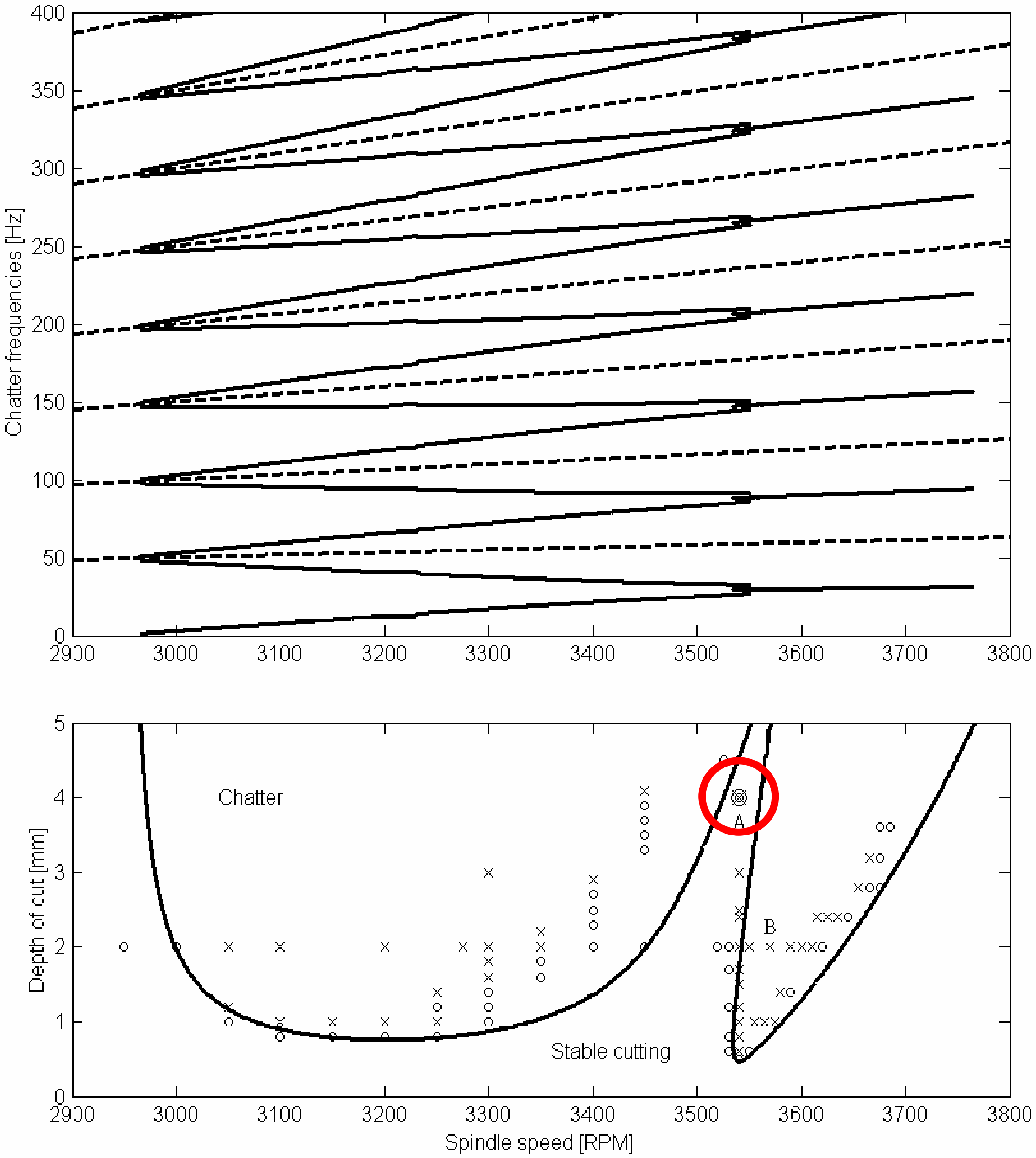


Lobes and lenses with ζ=0.02

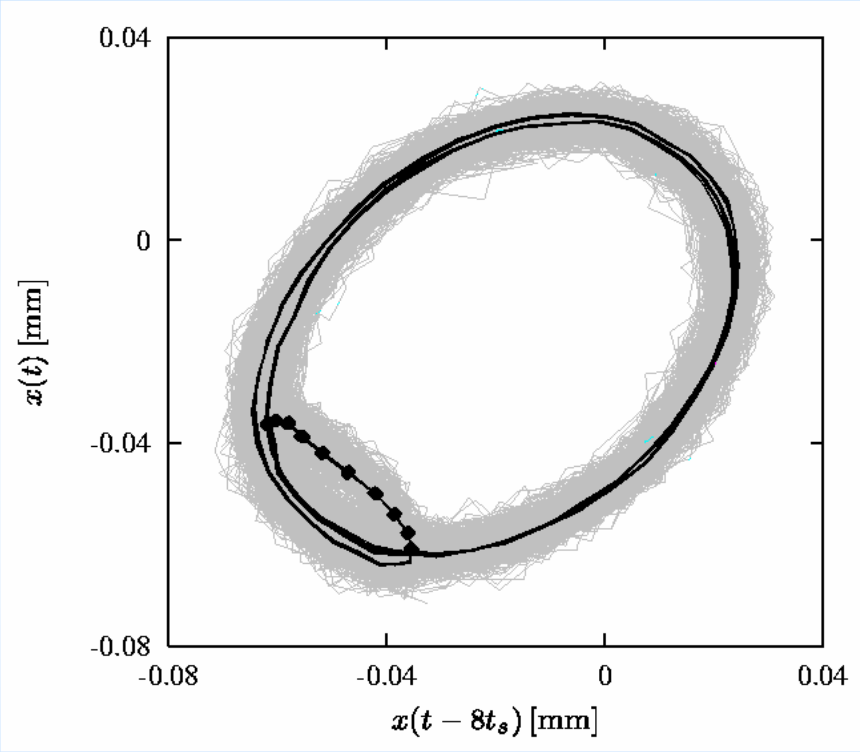


(Szalai,~, 2003)

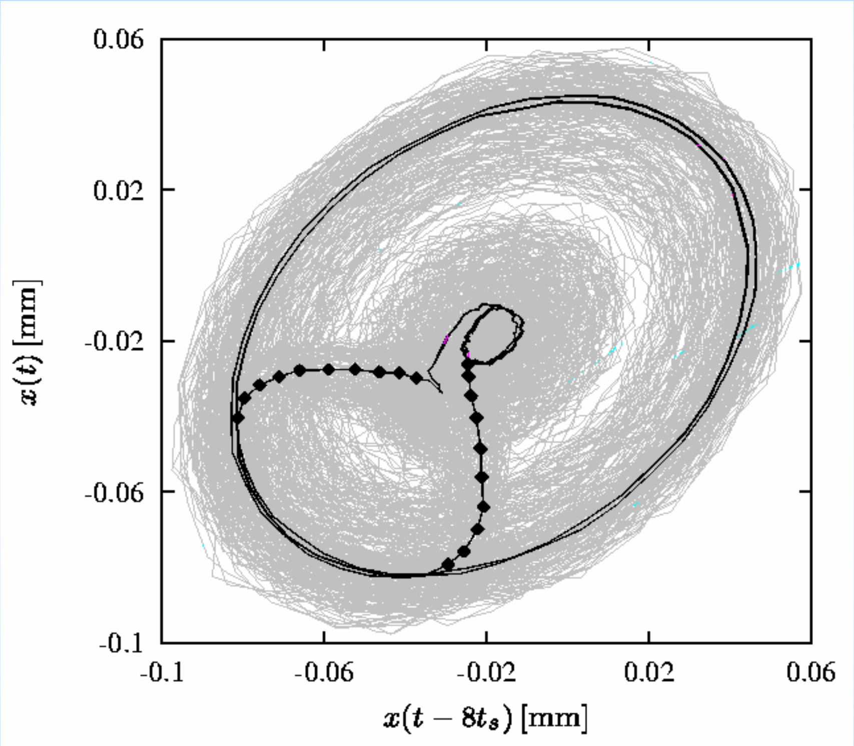
with ζ=0.0038



Phase space reconstruction at A



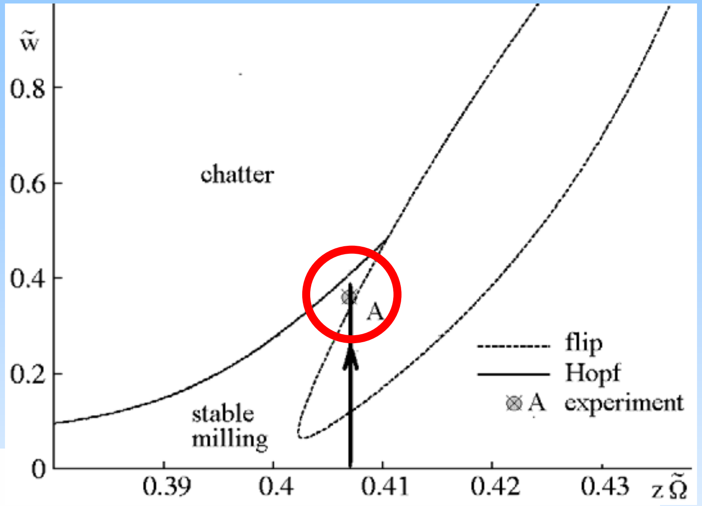
Stable milling

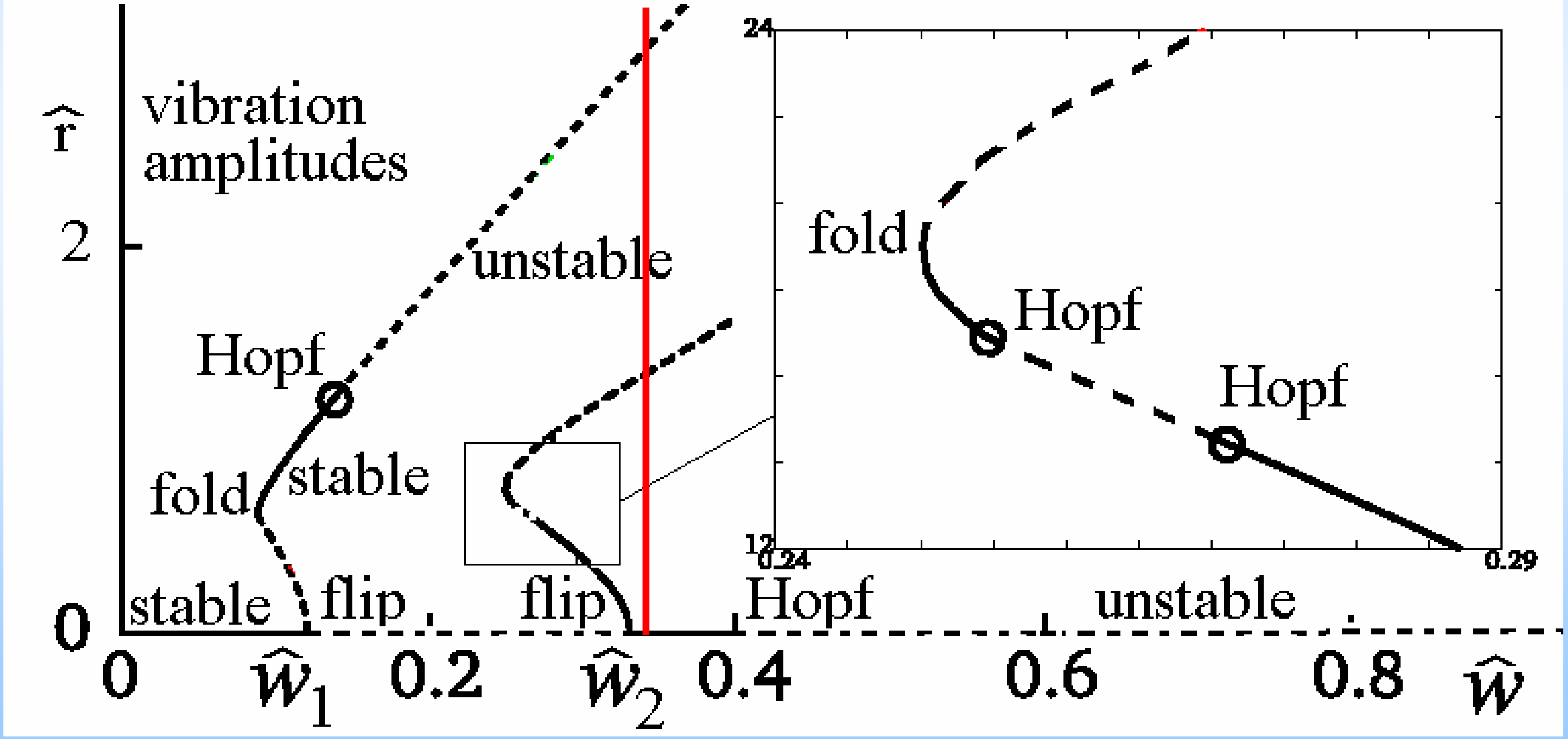


Unstable milling with stable period-2(?) or quasi-periodic(?) oscillation

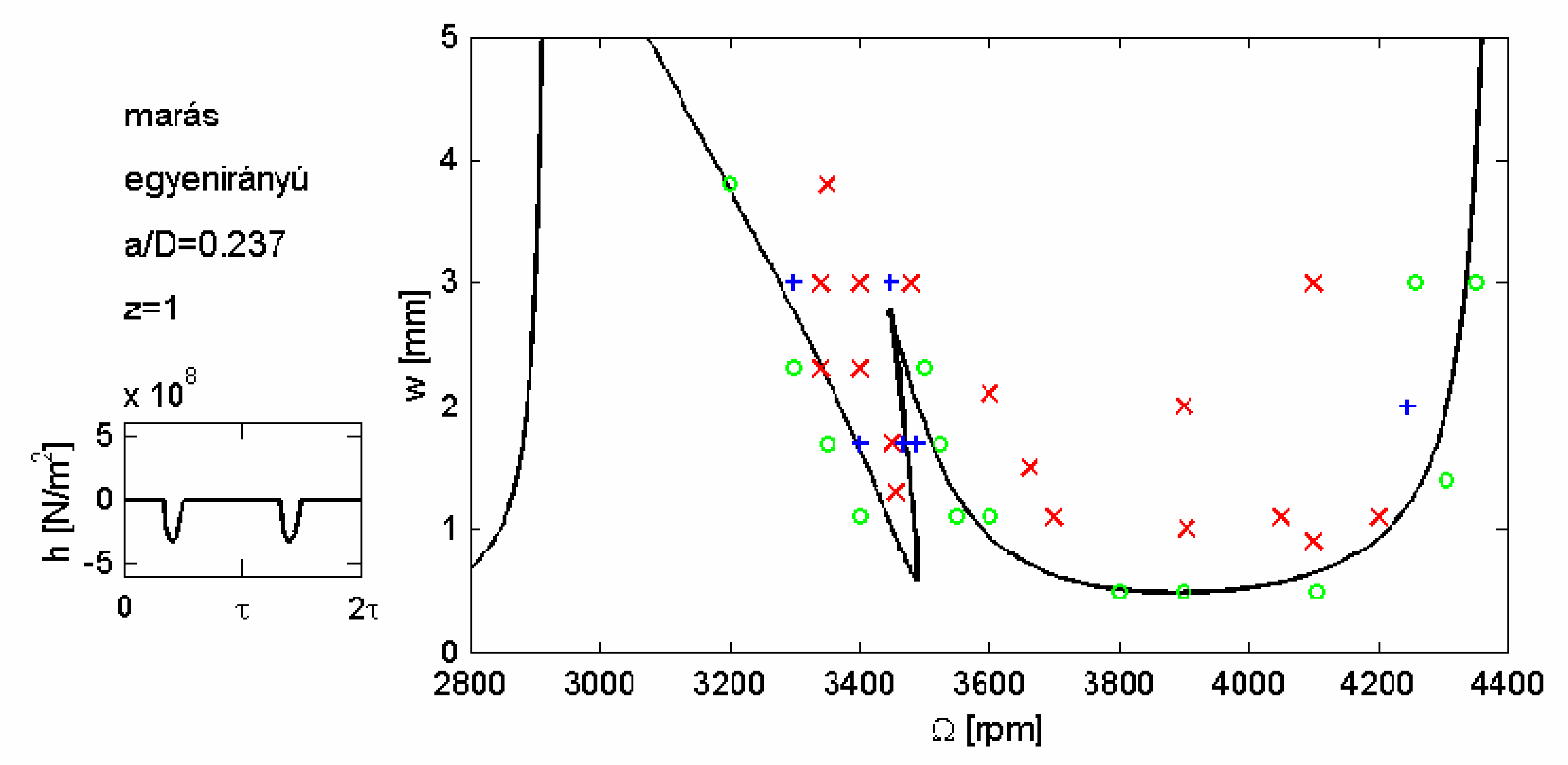
Bifurcation diagram

Kushke,Buckwar (2005) – Stochastic DDE





Stability of up- and down-milling



Stabilization by time-periodic parameters!

Insperger, Mann, ~, Bayly (2002)

Stabilization by time-periodic time delay

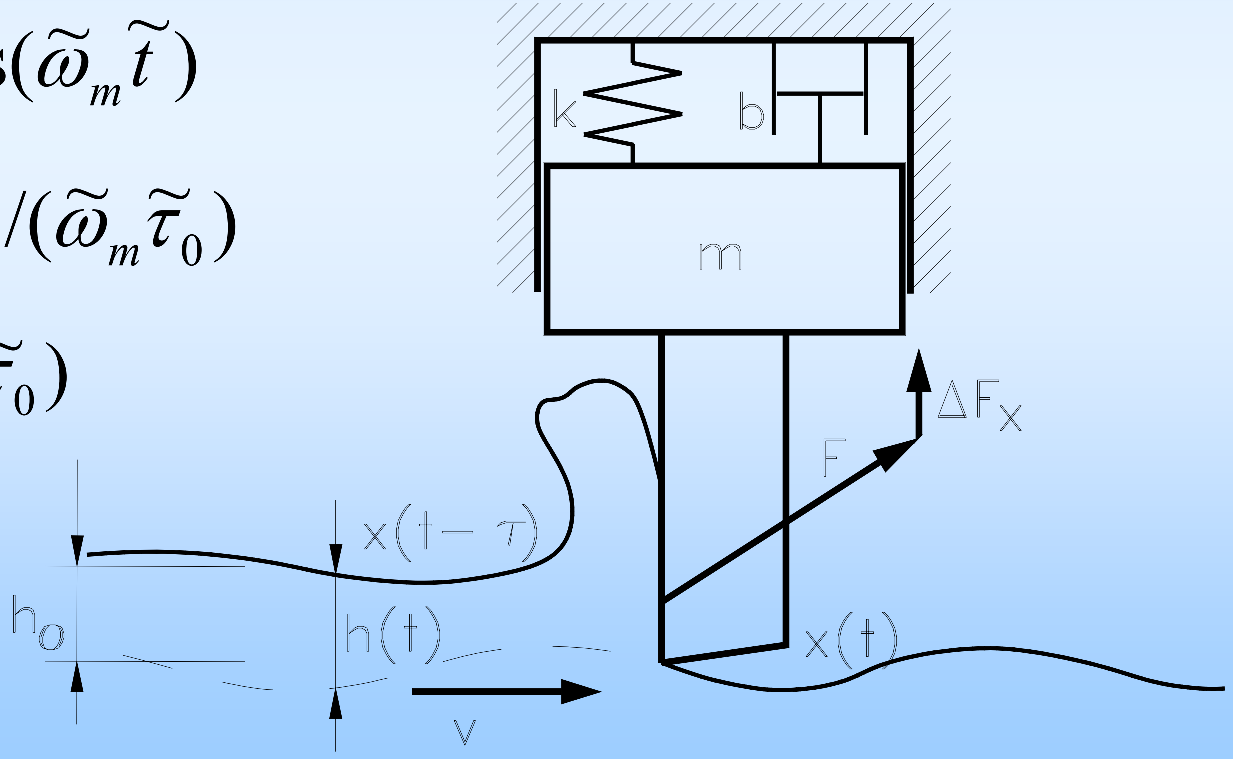
Chatter suppression by *spindle speed modulation*:

$$x''(\tilde{t})+2\zeta x'(\tilde{t})+(1+\tilde{w})x(\tilde{t})=\tilde{w}x(\tilde{t}-\tilde{\tau}(\tilde{t}))$$

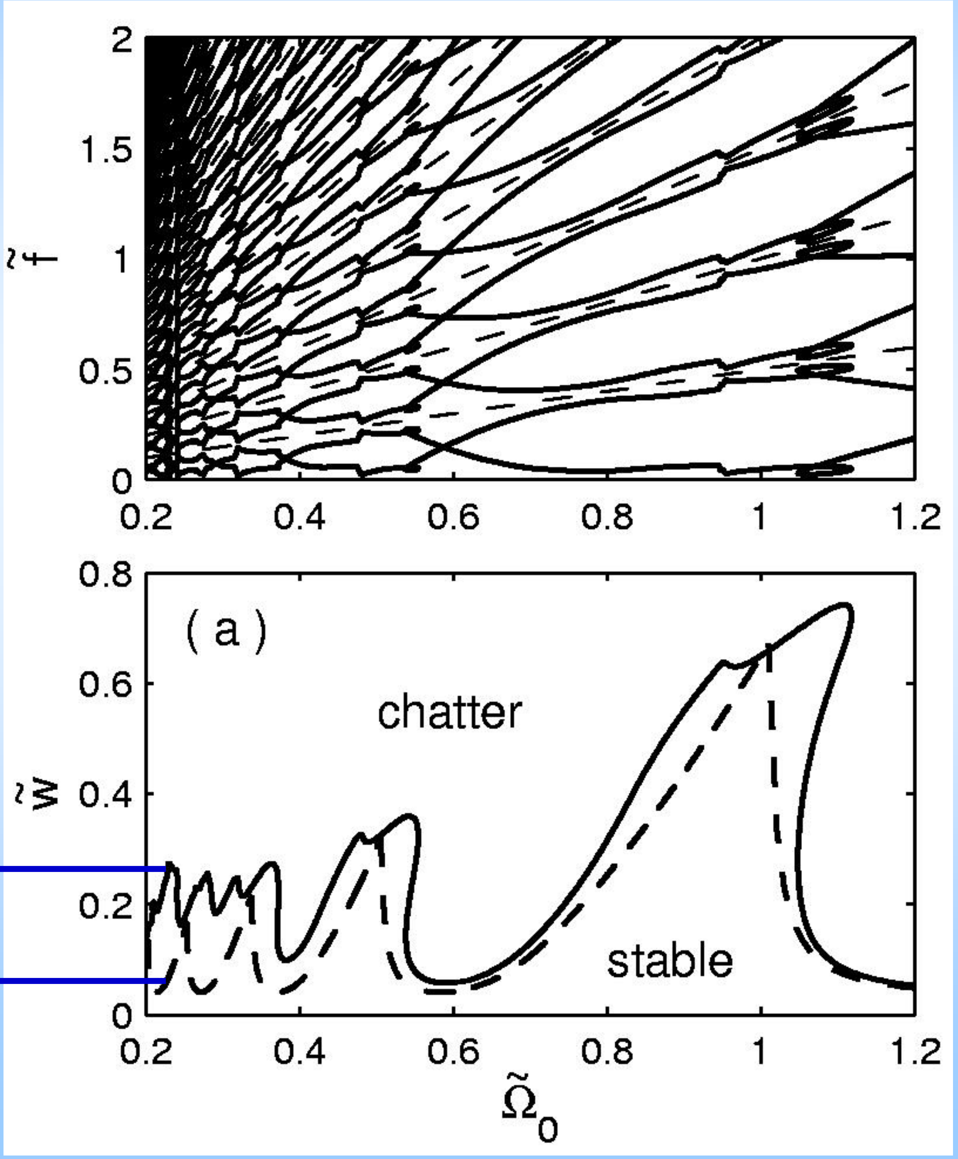
$$\tilde{\tau}(\tilde{t})=\tilde{\tau}_0+\tilde{\tau}_1\cos(\tilde{\omega}_m\tilde{t})$$

$$R_P=T_m/\tau_0=2\pi/(\tilde{\omega}_m\tilde{\tau}_0)$$

$$R_A=\tau_1/\tau_0=\tilde{\tau}_1/\tilde{\tau}_0)$$



Improved stability properties



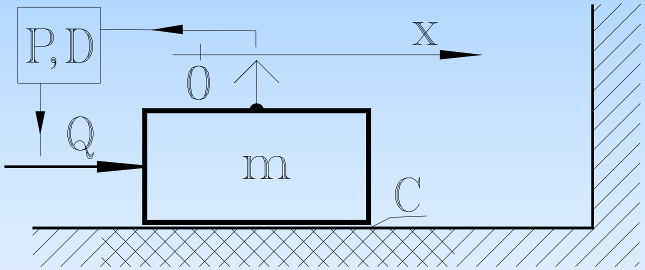
$R_P = 2$

(Hard to realize…)

$R_A = 0.1$

$R_A = 0$

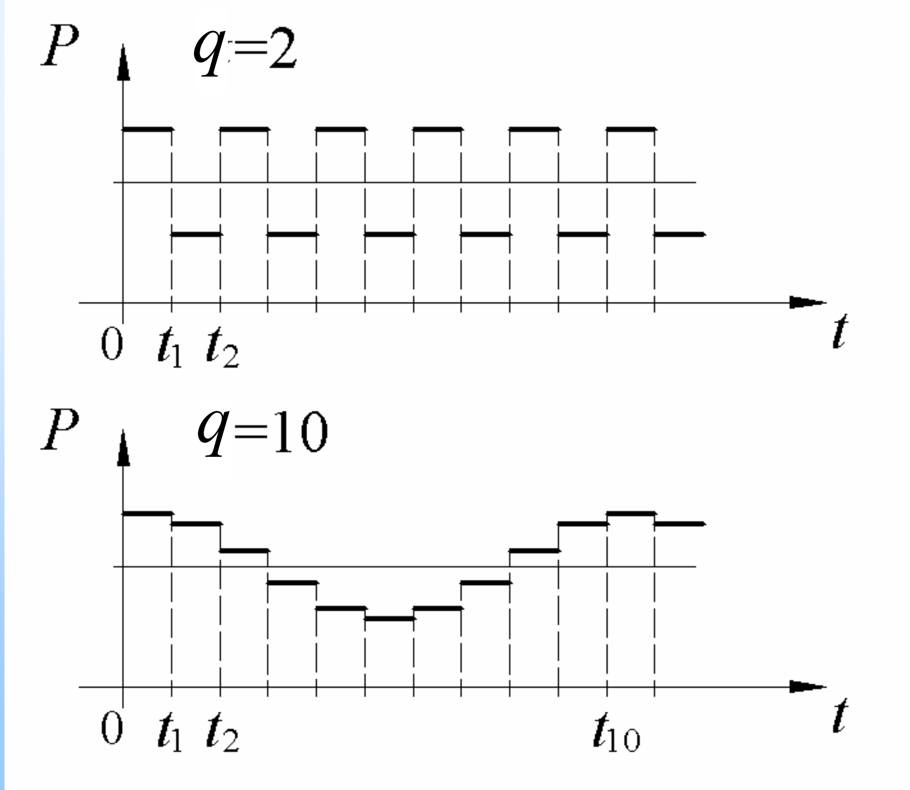
Time-periodic control gains



Period: $q\Delta t$ Delay: $r\Delta t$

$$P_{j+q} = P_j, \quad D_{j+q} = D_j$$

$$m\ddot{x}(t) = -P_j x((j-r)\Delta t) - D_j \dot{x}((j-r)\Delta t),$$

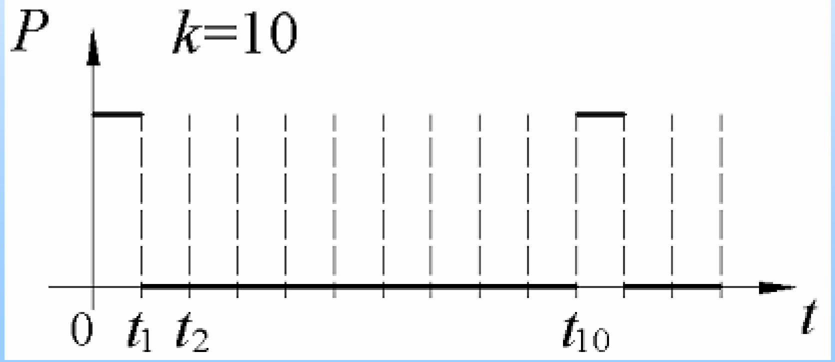


$$t \in [j\Delta t, (j+1)\Delta t),\ j = 0,1,2,\ldots$$

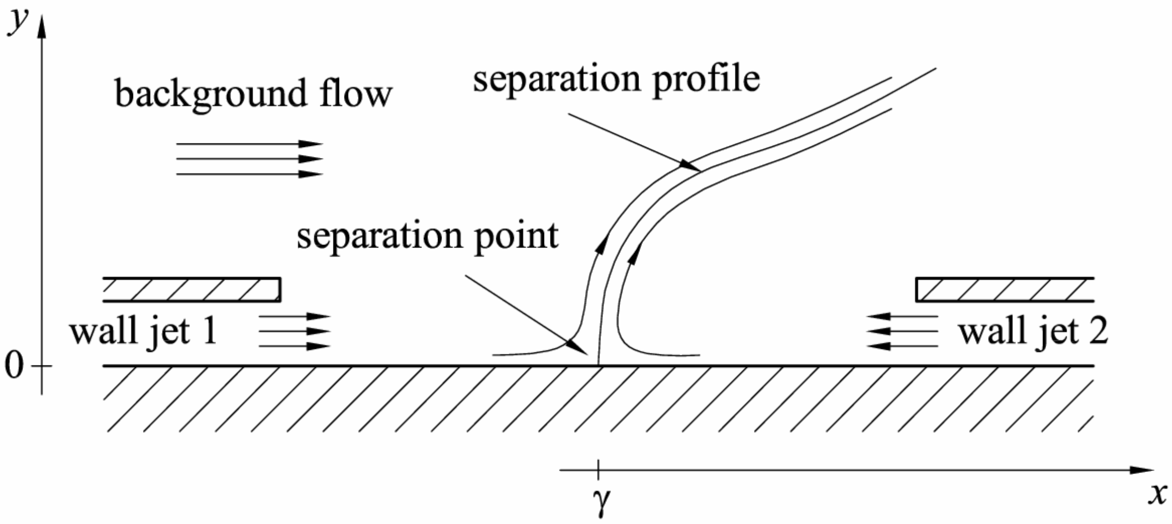
What periods to use?

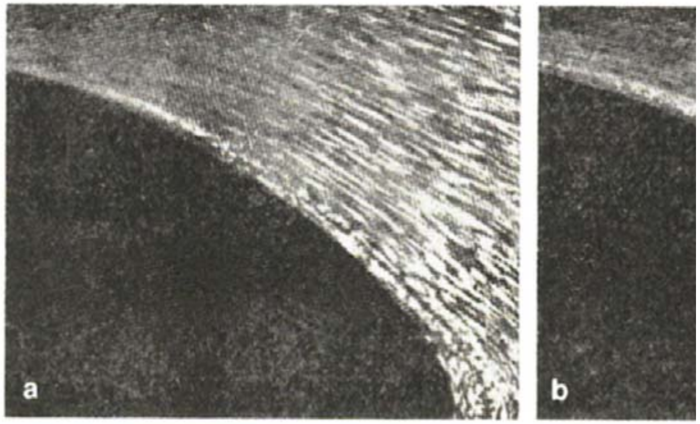
What functions to use?

Best: Act and wait!



Control of separation point in flow





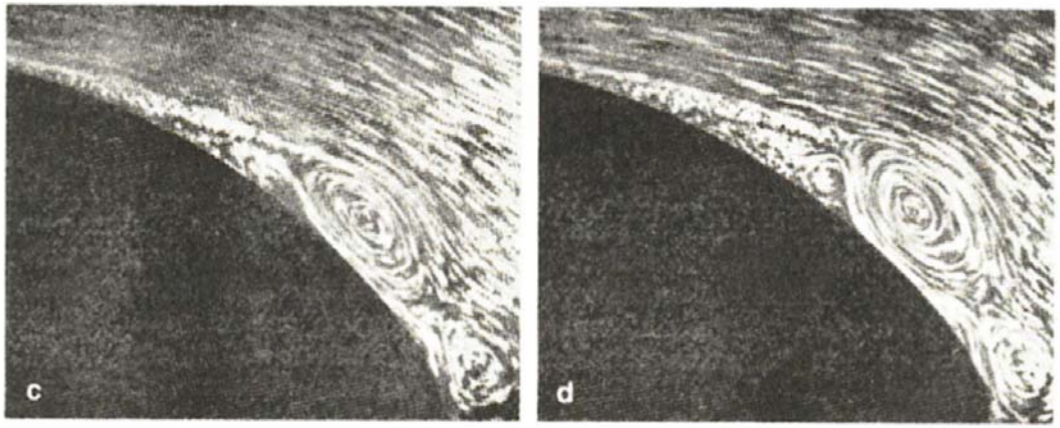
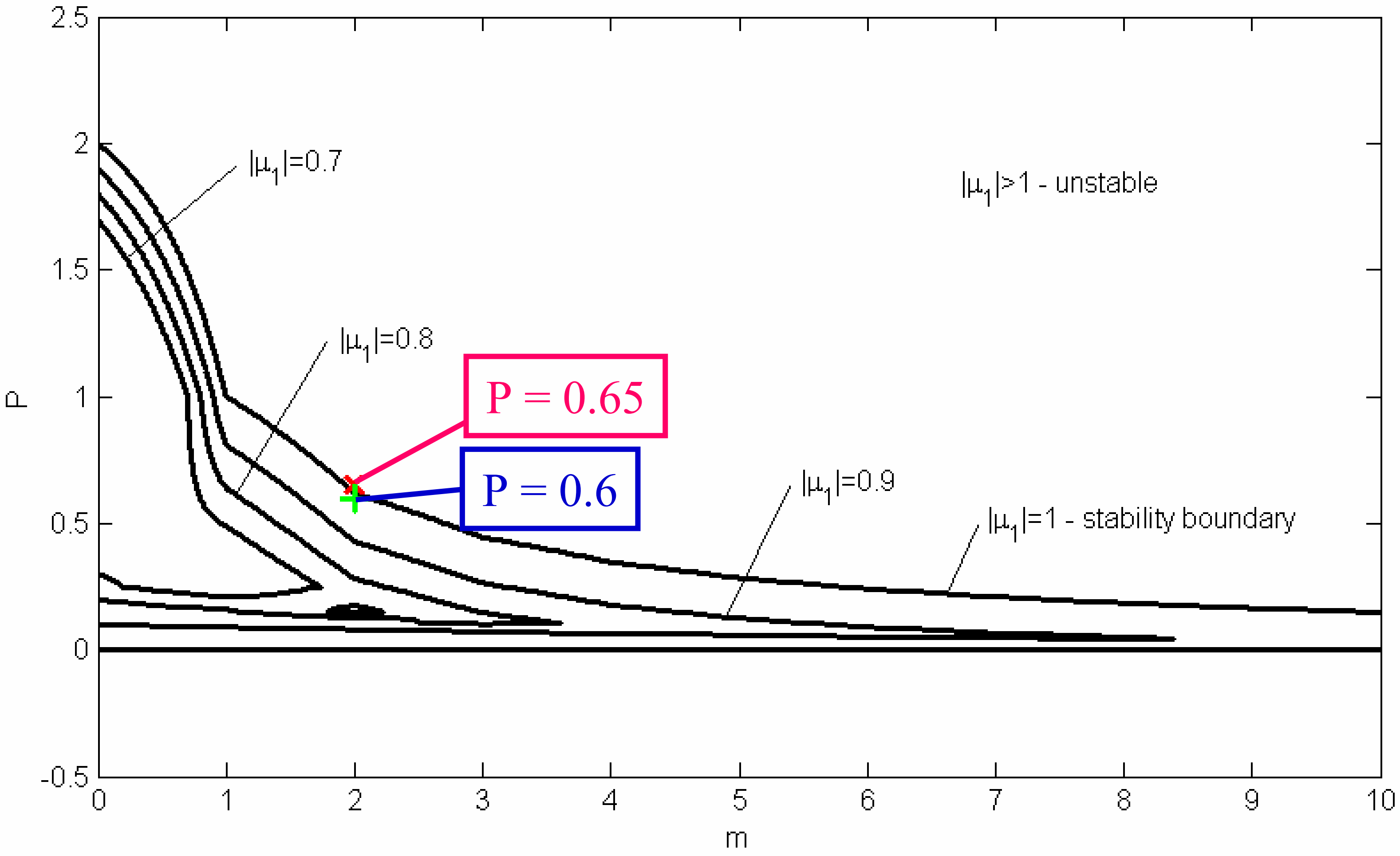


Fig. 2.7 a-d. Development in time of the separation at the back of a blunt body, after L. Prandtl; O. Tietjens (1931)

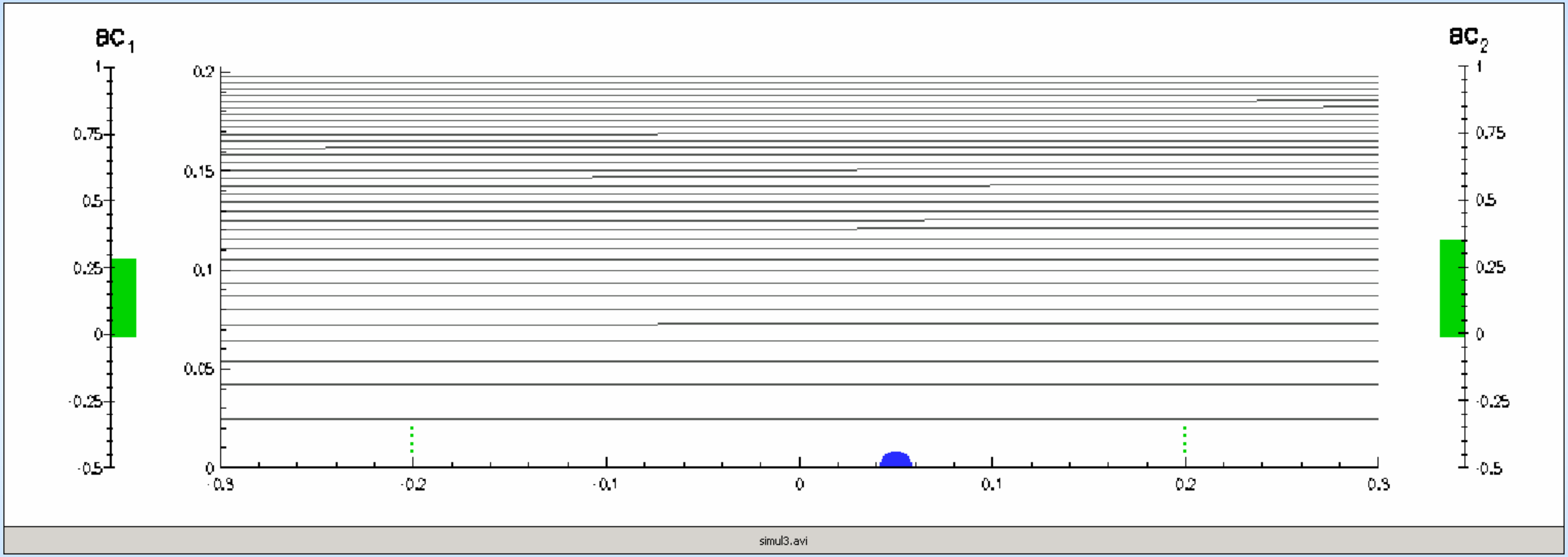
Separation criteria
(Haller, 2004)
fluid particles break away from the wall

Aim (in jet engines):
send fuel particles to specific prescribed locations

Stability chart

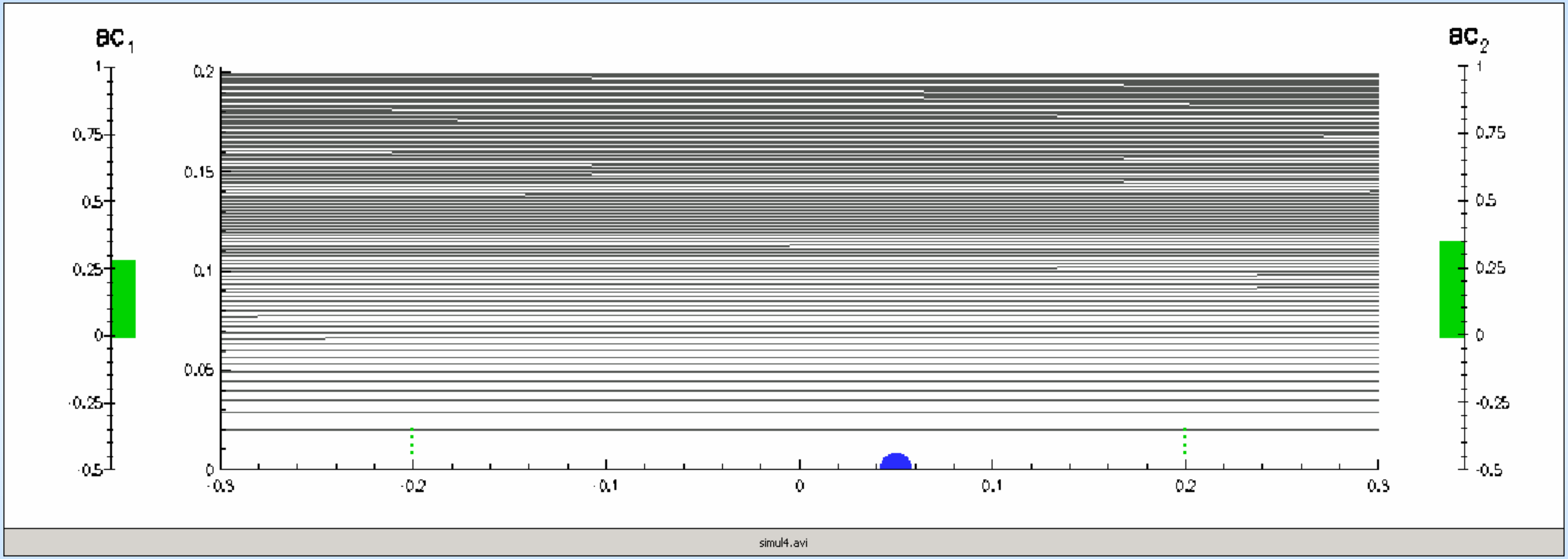


Simulation at $P = 0.6$

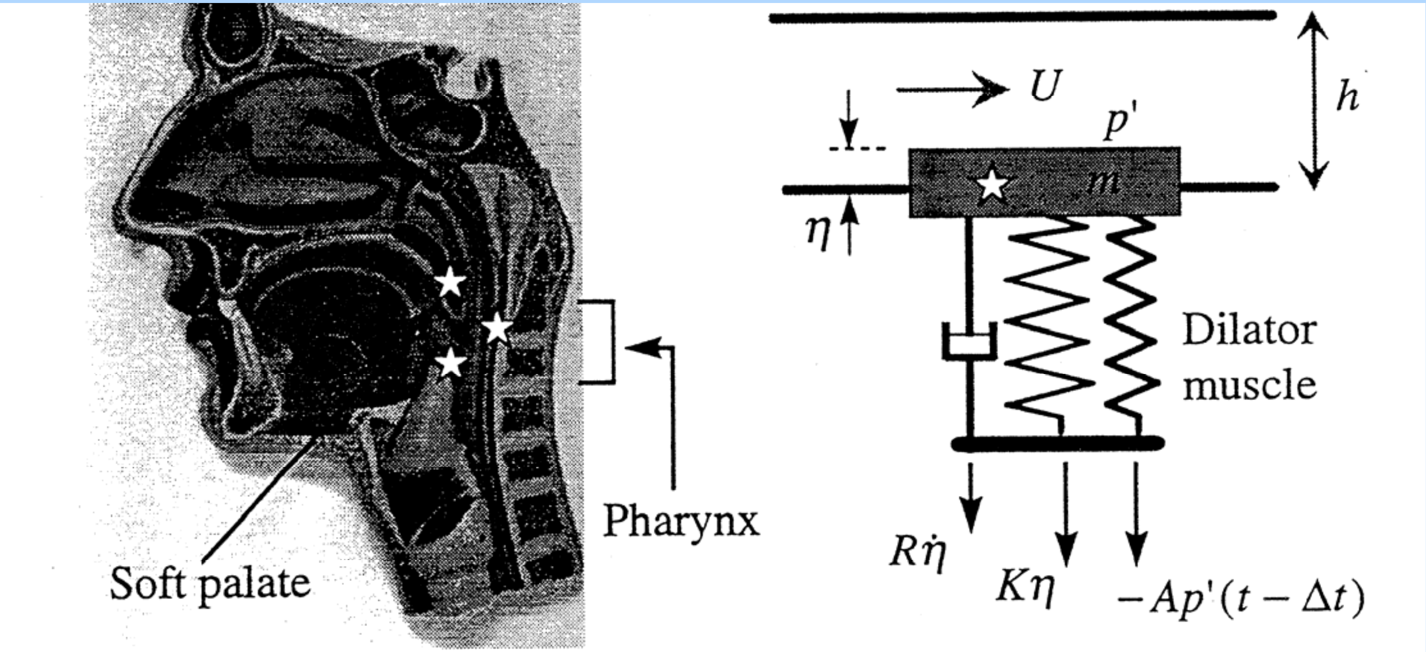


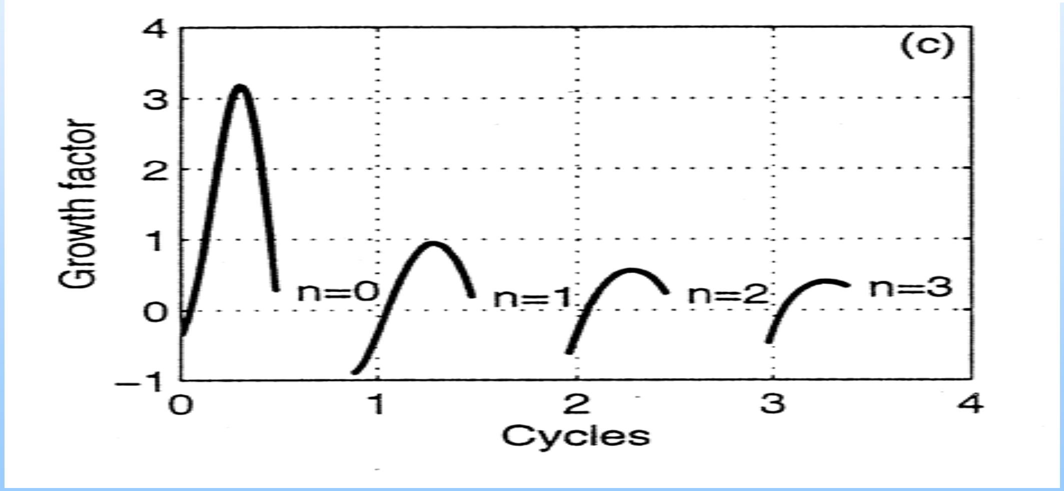
Lekien,Insperger,Salman,~,Haller (2004)

Simulation at $P = 0.65$



Snoring





Ffowcs

(Cambridge, 1997)

Conclusion

- Periodic modulation of gain parameters may result improvements in the stability of systems with large delay, but
- It may also cause loss of stability via period-2 oscillations, too
- Periodic gain variation just larger than the delay results improvements in stability and robustness (act and wait)

Thank you for your attention!