

Periodic motions and resonances in autonomous anisotropic central force fields

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Abstract

We are interested in the motion of a particle in a planar central force field with an anisotropic attracting force. In this framework, we provide some conditions for the existence, multiplicity, and localization of periodic solutions. In the case of the perturbed Kepler or harmonic potential we show how resonance can arise, even if the system is autonomous.

1 Introduction

This paper is concerned with the analysis of some aspects of the dynamics of a particle moving under the action of an anisotropic central force in $\mathbb{R}^3$. The force field is said to be *anisotropic* if it lacks rotational symmetry, i.e., it is not merely a function of the distance from the origin. In our case, the force is always assumed to be radial (i.e., directed toward the origin), so that anisotropy only affects its magnitude, and not its direction. Hence, when the mass is normalized, the motion of the particle is governed by the autonomous system

$$\ddot{\mathbf{x}} = -F(\mathbf{x}) \frac{\mathbf{x}}{|\mathbf{x}|}. \quad (1)$$

Since the angular momentum vector $\mathbf{x} \times \dot{\mathbf{x}}$ is conserved, every orbit is planar, and the analysis can be reduced to a two-dimensional setting. We therefore assume $F : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}$ and look for collisionless solutions $\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^2$, i.e.,

$$\mathbf{x}(t) \neq 0, \quad \dot{\mathbf{x}}(t) \neq 0,$$

for every t . It is convenient to introduce polar coordinates $\mathbf{x} = \rho e^{i\theta}$, where $\rho = |\mathbf{x}| > 0$ and $\theta = \arg \mathbf{x}$. Writing $F(\mathbf{x}) = F(\rho, \theta)$, with a slight abuse of notation, the force field is anisotropic precisely when F depends explicitly on θ . This situation creates a difficulty due to the fact that, unlike the classical rotationally symmetric case $F(\rho, \theta) \equiv \mathcal{F}_0(\rho)$, it is not possible to decouple the radial variable from the angular one.

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Our original motivation comes from the anisotropic Kepler problem

$$\ddot{\mathbf{x}} = -a(\theta) \frac{\mathbf{x}}{|\mathbf{x}|^3}. \quad (2)$$

Such a system may arise in several contexts. For instance, the body generating the gravitational field need not be perfectly spherical (see, e.g., [19] and the references therein). Another possible source of anisotropy is given by photogravitational effects arising from asymmetric radiation pressure. The force exerted by electromagnetic radiation on the surface of the particle may be significant, especially in highly luminous systems, such as those near stars. In particular, oblate or prolate stars – an occurrence that is typical of rapidly rotating stars – as well as limb darkening, may produce a photogravitational flux that varies with the polar angle, thereby breaking spherical symmetry [22]. Finally, anisotropic central potentials arise as well quite naturally in Molecular Dynamics [20].

Our results are of perturbative type. We consider the system

$$\ddot{\mathbf{x}} = -F(\rho, \theta; \varepsilon) \frac{\mathbf{x}}{|\mathbf{x}|}, \quad (S_\varepsilon)$$

where

$$F(\rho, \theta; \varepsilon) = \mathcal{F}_0(\rho) + \mathcal{G}(\rho, \theta; \varepsilon),$$

with

$$\mathcal{G}(\rho, \theta; 0) = 0.$$

Setting $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, we will assume $\mathcal{F}_0 : (0, +\infty) \rightarrow \mathbb{R}$ to be of class C^3 , while $\mathcal{G} : (0, +\infty) \times \mathbb{S}^1 \times (-1, 1) \rightarrow \mathbb{R}$ needs only be continuous. We will prove that, under some nondegeneracy assumptions, any periodic solution $\mathbf{x}_0(t)$ of the unperturbed system (S_0) gives rise to a multiplicity of periodic solutions of (S_ε) .

The bifurcation from circular orbits has been widely investigated, starting from the pioneering paper [3], based on an abstract result in [4] obtained by a variational technique. It is remarkable that the bifurcation from non-circular periodic solutions of rotationally symmetric central force problems has not been considered until recently, in [5, 6, 7], while perturbing the equation with a periodic time-dependent forcing term. Since the approach in these papers is variational, the perturbation term is assumed to be of the type $\nabla_{\mathbf{x}} U(t, \mathbf{x}; \varepsilon)$, i.e., the gradient of a certain function U with respect to the space variable. In this paper, our perturbation term $\mathcal{G}(\mathbf{x}; \varepsilon)$ is not assumed to be a gradient, hence the system does not appear to have a variational structure. However, since the system is autonomous, we can perform a Clairaut transformation and recover a Hamiltonian system where the Poincaré–Birkhoff theorem can be applied.

We obtain a bifurcation of subharmonic solutions from a manifold generated by a given periodic solution $\mathbf{x}_0(t) = \rho_0(t)e^{i\theta_0(t)}$ of the unperturbed system (see Theorems 8 and 9 for the precise statements.) Our main results thus stand alongside the ones obtained in [6, 7], following the research line started in [4].

The paper is organized as follows. In Section 2 we introduce our problem, where we assume $F(\mathbf{x}) = F(\rho, \theta; \varepsilon)$, and provide some preliminary information. Here ε is a small parameter, and $F(\rho, \theta; 0) = \mathcal{F}_0(\rho)$. Section 4 contains our main bifurcation results. The proofs make use of the Poincaré–Birkhoff Theorem as presented in [14], which we review in Section 3. Examples are given in Section 5 and 6, where in particular we show how to deal with power potentials, i.e., the case when $F(\rho, \theta; \varepsilon) = \rho^\alpha(c + \varepsilon b(\theta))$. Our results apply in the nonresonant case $\alpha \notin \{-2, 1\}$, while the cases $\alpha = -2$ (Kepler problem) and $\alpha = 1$ (harmonic oscillator) lead to possible resonances. Finally, in the Appendix we provide the proofs of two technical lemmas concerning the period function.

2 Some preliminaries

Let us consider (S_ε) and write the equivalent system

$$\ddot{\rho} - \rho \dot{\theta}^2 = -F(\rho, \theta; \varepsilon), \quad \rho \ddot{\theta} + 2\dot{\rho}\dot{\theta} = 0. \quad (3)$$

From the second equation it follows that the quantity

$$\ell = \rho^2 \dot{\theta},$$

to be referred to as *angular momentum*, is conserved along the motion. Without loss of generality, we will assume throughout the paper that $\ell > 0$. This means that our solutions rotate counter-clockwise around the origin.

A classical approach to the study of system (3) is based on the Clairaut transformation, as clarified in the following statement.

Proposition 1. *Let $\mathbf{x} = \rho e^{i\theta} : I \rightarrow \mathbb{R}^2 \setminus \{0\}$ be a solution of (S_ε) with angular momentum $\ell > 0$. Then the map $t \mapsto \theta(t)$ is invertible from I to $J = \theta(I)$ and, denoting by $\theta \mapsto t(\theta)$ its inverse, the function $u(\theta) = 1/\rho(t(\theta))$ solves the Binet–Clairaut equation*

$$u'' + u = \frac{F(1/u, \theta; \varepsilon)}{\ell^2 u^2}. \quad (BC_\varepsilon)$$

Vice versa, if $u : J \rightarrow (0, +\infty)$ is a solution of (BC_ε) , for some $\ell > 0$, then there exists a solution $\mathbf{x} = \rho e^{i\theta} : I \rightarrow \mathbb{R}^2 \setminus \{0\}$ of (S_ε) with angular momentum ℓ , such that $\theta(I) = J$ and $\rho(t) = 1/u(\theta(t))$, for every $t \in I$. In both cases,

$$\{\mathbf{x}(t) : t \in I\} = \left\{ \frac{e^{i\theta}}{u(\theta)} : \theta \in J \right\}.$$

Remark 2. *Referring to the notation in the above proposition, the following identities will be useful:*

$$\dot{\theta}(t) = \ell u^2(\theta(t)), \quad \dot{\rho}(t) = -\frac{1}{u^2(\theta(t))} u'(\theta(t)) \dot{\theta}(t) = -\ell u'(\theta(t)).$$

Moreover,

$$\dot{\mathbf{x}}(t) = \ell [-u'(\theta(t)) + iu(\theta(t))] e^{i\theta(t)}, \quad (4)$$

whence

$$|\dot{\mathbf{x}}(t)| = \ell \sqrt{u(\theta(t))^2 + u'(\theta(t))^2}. \quad (5)$$

Defining the function $g_\ell : (0, +\infty) \rightarrow \mathbb{R}$ by

$$g_\ell(u) = u - \frac{\mathcal{F}_0(1/u)}{\ell^2 u^2}, \quad (6)$$

equation (BC_0) reads as

$$u'' + g_\ell(u) = 0. \quad (7)$$

Here is how periodic solutions of (S_0) are related to those of (7).

Proposition 3. *Circular solutions $\mathbf{x}_0$ of (S_0) with radius $\rho_0 > 0$ correspond to equilibria u_0 of (7) through $u_0 = 1/\rho_0$. On the other hand, non-circular periodic solutions of (S_0) correspond to periodic solutions of (7) whose minimal period is commensurable with 2π .*

For a detailed proof of Propositions 1 and 3 see, e.g., [17, Ch. 3.5] or [15]. It is also useful to introduce the function $\mathcal{K}_\ell : (0, +\infty) \rightarrow \mathbb{R}$, defined as

$$\mathcal{K}_\ell(u) = 5[g_\ell''(u)]^2 - 3g_\ell'(u)g_\ell'''(u). \quad (8)$$

Definition 4. *We say that the pair $(u_0, \ell_0) \in \mathbb{R}^2$ is admissible if the following conditions hold:*

- (i) $u_0 > 0$ and $\ell_0 > 0$;
- (ii) $g_{\ell_0}(u_0) = 0$;
- (iii) $g'_{\ell_0}(u_0) > 0$;
- (iv) $\mathcal{K}_{\ell_0}(u_0) \neq 0$.

Notice that, if (u_0, ℓ_0) is admissible, then $u_0 > 0$ is an equilibrium of the equation

$$u'' + g_{\ell_0}(u) = 0, \quad (9)$$

by (ii), and $(u, v) = (u_0, 0)$ is a nondegenerate center for the equivalent system

$$u' = v, \quad v' = -g_{\ell_0}(u), \quad (10)$$

by (iii). Let us denote by $\mathcal{I}_0 = (0, \alpha_0)$ the maximal interval in $(0, +\infty)$ such that, if $a \in \mathcal{I}_0$, then the solution of system (10) with $u(0) = u_0 + a$, $v(0) = 0$ is periodic. For every $a \in \mathcal{I}_0$, let $\tau(a)$ be the minimal period of this solution. The function $\tau : \mathcal{I}_0 \rightarrow (0, +\infty)$ thus defined is usually called *period function*.

Our results are based on the following property.

Lemma 5. *If (u_0, ℓ_0) is admissible, then*

$$\tau(a) = \frac{2\pi}{\sqrt{g'_{\ell_0}(u_0)}} + \frac{\pi \mathcal{K}_{\ell_0}(u_0)}{24 g'_{\ell_0}(u_0)^{5/2}} a^2 + o(a^2) \quad \text{as } a \rightarrow 0^+.$$

The proof relies on the classical Lindstedt–Poincaré technique and is postponed to the Appendix.

Remark 6. *Let us mention that the formula defining the function $\mathcal{K}_\ell$ was used in [25] in order to study the monotonicity of the period map for more general planar Hamiltonian systems.*

3 The abstract bifurcation theorem

In this section we review a result from [14] which will be useful in the sequel. We start by considering a *period annulus* $\mathcal{A} \subseteq \mathbb{R}^2$ for an autonomous Hamiltonian system

$$J\dot{z} = \nabla \mathcal{H}_0(z), \quad (11)$$

i.e., a connected set in the plane covered by orbits of nonconstant periodic solutions. Here J is the standard 2×2 symplectic matrix, namely $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and the twice continuously differentiable Hamiltonian function $\mathcal{H}_0$ needs to be only defined on $\mathcal{A}$. We assume that $\mathcal{A}$ surrounds a given point z_0 , not belonging to $\mathcal{A}$ itself.

All the solutions with initial point in $\mathcal{A}$ are periodic and their orbits are not contractible in $\mathcal{A}$. Moreover, we assume that $\mathcal{A}$ is *not isochronous*, i.e., the periods of the solutions in $\mathcal{A}$ cover an interval $[\mathcal{T}_{\min}, \mathcal{T}_{\max}]$, with $\mathcal{T}_{\min} < \mathcal{T}_{\max}$.

We then consider the Hamiltonian system

$$J\dot{z} = \nabla_z H(t, z; \varepsilon), \quad (12)$$

whose Hamiltonian function $H : \mathbb{R} \times \mathcal{A} \times (-1, 1) \rightarrow \mathbb{R}$ is T -periodic in its first variable, for some $T > 0$, having a continuous gradient with respect to its second variable, denoted by $\nabla_z H(t, z; \varepsilon)$. We assume that

$$H(t, z; 0) = \mathcal{H}_0(z), \quad \text{for every } t \in [0, T] \text{ and } z \in \mathcal{A}.$$

Here is the result stated in [14, Theorem 2.7].

Theorem 7. *Given two positive integers m and n such that*

$$\mathcal{T}_{\min} < \frac{mT}{n} < \mathcal{T}_{\max},$$

there is an $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, then system (12) has at least two mT -periodic solutions, whose orbits are contained in $\mathcal{A}$, which make exactly n rotations around z_0 in the period time mT .

Notice that in [14] the gradient of H was also assumed to be locally Lipschitz continuous, in order to guarantee the uniqueness of the solutions to initial value problems. However, in view of [16], this assumption is not really needed (see also [13, Theorem 3.2]).

4 Main results

Our first result can be viewed as a bifurcation from a circular solution of (S_0) .

Theorem 8. *Let $\rho_0 > 0$ and $\ell_0 > 0$ be such that, setting $u_0 = 1/\rho_0$, the pair (u_0, ℓ_0) is admissible. Then, for every $\sigma > 0$ and any positive integer N , there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, system (S_ε) has at least N periodic solutions $\mathbf{x}(t)$, with angular momentum ℓ_0 , satisfying*

$$\rho_0 - \sigma \leq |\mathbf{x}(t)| \leq \rho_0 + \sigma, \quad \frac{\ell_0}{\rho_0} - \sigma \leq |\dot{\mathbf{x}}(t)| \leq \frac{\ell_0}{\rho_0} + \sigma, \quad (13)$$

for every $t \in \mathbb{R}$.

Proof. Fix $\sigma > 0$, $N \in \mathbb{N} \setminus \{0\}$, and let $\mathbf{x}_0(t)$ be a circular solution of (S_0) with radius $\rho_0 > 0$ and angular momentum $\ell_0 > 0$, provided by Proposition 1. The equilibrium $z_0 := (u_0, 0)$ of the corresponding Binet–Clairaut system (10) is a center and, by Lemma 5, we can choose two nearby periodic solutions of system (10) with respective minimal periods $\Theta_1 < \Theta_2$, whose orbits bound a period annulus $\mathcal{A}$, with

$$\delta := \max\{d(z, z_0) : z \in \mathcal{A}\} < \min\left\{\frac{1}{2\rho_0}, \frac{\sigma}{2\rho_0^2}, \frac{\sigma}{\ell_0}\right\}.$$

We then select N numbers of the type $2\pi m_k/n_k \in 2\pi\mathbb{Q}$ in the interval (Θ_1, Θ_2) , with $k = 1, \dots, N$.

Notice that, setting $z = (u, v)$ and choosing G_ℓ to be a primitive of g_ℓ , system (10) can be written in the form (11), with Hamiltonian function $\mathcal{H}(z) = \frac{1}{2}v^2 + G_{\ell_0}(u)$. At the same time we can write equation (BC_ε) with $\ell = \ell_0$ as an equivalent system

$$u' = v, \quad v' = -g_{\ell_0}(u) + \frac{\mathcal{G}(1/u, \theta; \varepsilon)}{\ell_0^2 u^2},$$

which is a Hamiltonian system of the type (12). Therefore, applying Theorem 7, we deduce that, when $|\varepsilon|$ is small, for each selected pair (m_k, n_k) there are at least two $2\pi m_k$ -periodic solutions $u_k(\theta)$ of (BC_ε) which make exactly n_k rotations around z_0 in their period time,

whose orbits lie in $\mathcal{A}$. For any of these solutions $u(\theta)$, we have that $|u(\theta) - u_0| < \delta$ and, since $\delta < u_0/2$ it follows that $u(\theta) > u_0/2$ and hence

$$\left| \frac{1}{u(\theta)} - \frac{1}{u_0} \right| = \frac{|u(\theta) - u_0|}{u(\theta)u_0} \leq \frac{2\delta}{u_0^2} < \sigma. \quad (14)$$

We now recover, by Proposition 1, the N periodic solutions $\mathbf{x}_k(t)$ we are looking for. Indeed, deleting for simplicity the indices, since $|\mathbf{x}(t)| = 1/u(\theta(t))$, the first estimate in (13) follows directly from (14). Then, since $(u(\theta), u'(\theta)) \in \mathcal{A}$, we have that

$$u_0 - \delta \leq \sqrt{u(\theta(t))^2 + u'(\theta(t))^2} \leq u_0 + \delta,$$

hence, recalling (5),

$$\left| \frac{|\dot{\mathbf{x}}(t)|}{\ell_0} - u_0 \right| \leq \delta,$$

which leads to the second estimate in (13), since $\delta < \sigma/\ell_0$. Finally, from the $2\pi m$ -periodicity of u and the relation

$$\frac{dt}{d\theta} = \frac{1}{\ell_0 u(\theta)^2},$$

we see that $\mathbf{x}$ is periodic, with period

$$T = \int_0^{2\pi m} \frac{ds}{\ell_0 u(s)^2}.$$

The proof is thus completed. $\square$

We are now interested in bifurcating from *any* periodic solution of (S_0) , not necessarily a circular one. However, in order to achieve this task, we will need to assume the function $\mathcal{F}_0$ to be analytic. Let us first notice that, if we fix a periodic solution $\mathbf{x}_0(t)$ of (S_0) with angular momentum ℓ_0 and write $\mathbf{x}_0(t) = \rho_0(t)e^{i\theta_0(t)}$, then for every $\omega \in \mathbb{R}$ also the rotated curve

$$\mathbf{x}_\omega(t) := e^{i\omega} \mathbf{x}_0(t) = \rho_0(t)e^{i(\theta_0(t)+\omega)}$$

is a periodic solution of (S_0) , and all these solutions have the same angular momentum. Moreover, if $u_*(\theta)$ denotes the periodic solution of the Binet–Clairaut equation (9) associated with $\mathbf{x}_0$, then the solution associated with $\mathbf{x}_\omega$ is $u_\omega(\theta) = u_*(\theta - \omega)$. In particular, all these solutions of (9) have the same orbit.

Theorem 9. *Let (u_0, ℓ_0) be an admissible pair and suppose that*

$$g_{\ell_0}(u) = 0 \quad \Longleftrightarrow \quad u = u_0. \quad (15)$$

Assume moreover that $\mathcal{F}_0 : (0, +\infty) \rightarrow \mathbb{R}$ is analytic. Consider a (possibly non-circular) periodic solution $\mathbf{x}_0(t)$ of (S_0) with angular momentum ℓ_0 . Define

$$\mathcal{M} = \{(e^{i\omega} \mathbf{x}_0(s), e^{i\omega} \dot{\mathbf{x}}_0(s)) : \omega, s \in \mathbb{R}\}.$$

Then, for every $\sigma > 0$ and every positive integer N , there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, system (S_ε) has at least N periodic solutions $\mathbf{x}(t)$, with angular momentum ℓ_0 , satisfying

$$\text{dist} \left((\mathbf{x}(t), \dot{\mathbf{x}}(t)), \mathcal{M} \right) \leq \sigma, \quad \text{for every } t \in \mathbb{R}. \quad (16)$$

Notice that assumption (15) means that there is a unique circular orbit of (S_0) having angular momentum ℓ_0 .

For the proof we need some preliminaries. Let us consider the scalar equation

$$x'' + V'(x) = 0, \quad (17)$$

where V is a regular function defined on an open interval containing 0, such that $V(0) = V'(0) = 0$ and $V''(0) > 0$. Under these assumptions the origin is a center for the corresponding Hamiltonian system

$$x' = y, \quad y' = -V'(x), \quad (18)$$

with Hamiltonian function $H(x, y) = \frac{1}{2}y^2 + V(x)$. Let $\mathcal{J}_0 = (0, \beta_0)$ be the maximal interval in $(0, +\infty)$ such that, if $h \in \mathcal{J}_0$, the curve $H(x, y) = h$ corresponds to a periodic solution of system (18), and denote by $T(h)$ the minimal period of this solution.

Lemma 10. *If V is analytic, then the function $T : \mathcal{J}_0 \rightarrow (0, +\infty)$ is analytic, as well.*

The proof of this lemma is rather standard; we will however provide it in the Appendix, for the reader's convenience.

Proof of Theorem 9. Let $\mathbf{x}_0(t)$ be the solution of (S_0) in the statement, with angular momentum $\ell_0 > 0$. Writing $\mathbf{x}_0(t) = \rho_0(t)e^{i\theta_0(t)}$, correspondingly we have a periodic solution $u_*(\theta)$ of (9). Its orbit in the phase plane surrounds an equilibrium which, by the uniqueness assumption (15), is precisely u_0 . Since (u_0, ℓ_0) is admissible, this equilibrium generates a period annulus and, since there are no other equilibria, the orbit of u_* must belong to this period annulus.

We write $u_*(\theta) = u_0 + x(\theta)$ and define

$$V(x) = \int_0^x g_{\ell_0}(u_0 + s) ds,$$

so that x solves (17). Notice that, since $\mathcal{F}_0$ is analytic, by (6) also g_{ℓ_0} is analytic, and such is also V . Hence, by Lemma 10, the function $T : \mathcal{J}_0 \rightarrow (0, +\infty)$ is analytic. Given $a \in \mathcal{I}_0$, let u be the solution of (9) satisfying $u(0) = u_0 + a$, $u'(0) = 0$. If we now define

$$h_a = \frac{1}{2}x'(0)^2 + V(x(0)) = V(a),$$

we have that $\tau(a) = T(h_a) = T(V(a))$, thus showing that the period function τ is also analytic. Lemma 5 shows that the period function is not constant near the center, hence, by analyticity, it cannot be constant on any nontrivial interval.

Let

$$\Gamma_* = \{(u_*(\theta), u'_*(\theta)) : \theta \in \mathbb{R}\}.$$

The analyticity and nonconstancy of the period function imply that, near Γ_* , we can choose two periodic solutions of system (10) with respective minimal periods $\Theta_1 < \Theta_2$, whose orbits bound an arbitrarily thin period annulus $\mathcal{A}$. We then select N numbers of the type $2\pi m_k/n_k \in 2\pi\mathbb{Q}$ in the interval (Θ_1, Θ_2) , with $k = 1, \dots, N$. Hence, applying Theorem 7, for $|\varepsilon|$ sufficiently small we obtain N periodic solutions u_k of the perturbed Binet–Clairaut equation whose orbits are contained in $\mathcal{A}$. As in the proof of Theorem 8, we recover the periodic solutions $\mathbf{x}_k$ of (S_ε) by Proposition 1, with $k = 1 \dots, N$.

We now need to prove that closeness to Γ_* in the Binet–Clairaut phase plane implies closeness to $\mathcal{M}$ in the original phase space. Let

$$m_* = \min_{\theta \in \mathbb{R}} u_*(\theta) > 0,$$

fix $\eta > 0$ such that $\eta < m_*/2$, and choose $\mathcal{A}$ so thin that

$$\text{dist}((u_k(\theta), u'_k(\theta)), \Gamma_*) < \eta,$$

for every $\theta \in \mathbb{R}$ and every $k = 1, \dots, N$.

Choose one of the solutions u_k , e.g., u_1 ; the corresponding solution is $\mathbf{x}_1(t) = \frac{1}{u_1(\theta(t))} e^{i\theta_1(t)}$. Fix $t \in \mathbb{R}$ arbitrarily and set $\theta = \theta_1(t)$. There exists $\varphi \in \mathbb{R}$ such that

$$|(u_1(\theta), u'_1(\theta)) - (u_*(\varphi), u'_*(\varphi))| < \eta.$$

Since the angular momentum of $\mathbf{x}_0$ is positive, θ_0 is strictly increasing, hence there is a unique $s \in \mathbb{R}$ such that $\theta_0(s) = \varphi$. Set $\omega = \theta - \varphi$. Using Proposition 1, we can write

$$e^{i\omega} \mathbf{x}_0(s) = e^{i\omega} \frac{e^{i\varphi}}{u_*(\varphi)} = \frac{e^{i\theta}}{u_*(\varphi)},$$

hence,

$$|\mathbf{x}_1(t) - e^{i\omega} \mathbf{x}_0(s)| = \left| \frac{1}{u_1(\theta)} - \frac{1}{u_*(\varphi)} \right|.$$

Since

$$u_*(\varphi) \geq m_*, \quad u_1(\theta) \geq \frac{m_*}{2},$$

we obtain

$$|\mathbf{x}_1(t) - e^{i\omega} \mathbf{x}_0(s)| \leq \frac{2}{m_*^2} |u_1(\theta) - u_*(\varphi)| \leq \frac{2\eta}{m_*^2}.$$

Moreover, recalling (4),

$$\dot{\mathbf{x}}_1(t) = \ell_0 [-u'_1(\theta) + iu_1(\theta)] e^{i\theta},$$

while

$$e^{i\omega} \dot{\mathbf{x}}_0(s) = \ell_0 [-u'_*(\varphi) + iu_*(\varphi)] e^{i\theta}.$$

Therefore,

$$\left| \dot{\mathbf{x}}_1(t) - e^{i\omega} \dot{\mathbf{x}}_0(s) \right| = \ell_0 \sqrt{|u'_1(\theta) - u'_*(\varphi)|^2 + |u_1(\theta) - u_*(\varphi)|^2} < \ell_0 \eta.$$

Consequently,

$$\text{dist}((\mathbf{x}_1(t), \dot{\mathbf{x}}_1(t)), \mathcal{M}) \leq C\eta,$$

where $C > 0$ is independent of t . Since the period annulus $\mathcal{A}$ can be chosen arbitrarily thin, η can be taken arbitrarily small. Choosing it so that $C\eta < \sigma$ gives (16), and the proof is thus completed. $\square$

Remark 11. Notice that, in the circular case, the rotation $\mathbf{x}_\omega$ of the orbit $\mathbf{x}_0$ preserves the image, which is a circle. This explains why in the statement of Theorem 8 we simply bifurcate from the circle itself.

5 A perturbed power-type central force

We now apply the above results to the case

$$F(\rho, \theta; \varepsilon) = \rho^\alpha (c + \varepsilon b(\theta)),$$

where $\alpha \in \mathbb{R}$, $c > 0$, and $b : \mathbb{S}^1 \rightarrow \mathbb{R}$ is a continuous function. We assume $|\varepsilon|$ small enough so to have $c + \varepsilon b(\theta) > 0$ for every θ . System (S_ε) reads as

$$\ddot{\mathbf{x}} = -(c + \varepsilon b(\theta)) |\mathbf{x}|^{\alpha-1} \mathbf{x}, \tag{S_\varepsilon}$$

and equation (BC_ε) becomes

$$u'' + u = \frac{c + \varepsilon b(\theta)}{\ell^2} u^{-(\alpha+2)}. \tag{19}$$

In this case,

$$g_\ell(u) = u - \frac{c}{\ell^2} u^{-(\alpha+2)},$$

hence equation (7) reads as

$$u'' + u = \frac{c}{\ell^2} u^{-(\alpha+2)}, \tag{20}$$

and

$$\mathcal{K}_\ell(u) = \frac{c}{\ell^2} (\alpha + 2)(\alpha + 3) u^{-(\alpha+5)} \left(\frac{c}{\ell^2} u^{-(\alpha+3)} (\alpha + 2)(2\alpha + 3) - 3(\alpha + 4) \right).$$

A straightforward computation shows that if $\alpha = -2$, or $\alpha = 1$, there are no admissible pairs (u_0, ℓ_0) , hence our theorems cannot be applied. In these cases it is well-known that equation (20) has an isochronous dynamics. We postpone the analysis of these resonant cases to the end of the section.

Before stating our bifurcation result, let us describe the dynamics of the solutions of (20). It corresponds to a conservative system with energy

$$h = \frac{1}{2}(u')^2 + G_\ell(u),$$

where we can take

$$G_\ell(u) = \begin{cases} \frac{1}{2}u^2 + \frac{c}{\ell^2(\alpha+1)} u^{-(\alpha+1)}, & \alpha \neq -1, \\ \frac{1}{2}u^2 - \frac{c}{\ell^2} \log u, & \alpha = -1. \end{cases}$$

If $\alpha \neq -3$, for any $\ell_0 > 0$ there is a unique equilibrium of (20), namely

$$u_0 = \left(\frac{c}{\ell_0^2} \right)^{\frac{1}{\alpha+3}}. \quad (21)$$

If $\alpha > -3$, the equilibrium is a center and the small-amplitude period satisfies

$$T(h) \rightarrow \frac{2\pi}{\sqrt{\alpha+3}} \quad \text{as } h \rightarrow G_\ell(u_0)^+.$$

Any periodic orbit oscillates between turning points $u_1(h) < u_2(h)$ solving $G_\ell(u) = h$, and its period is

$$T(h) = \sqrt{2} \int_{u_1(h)}^{u_2(h)} \frac{du}{\sqrt{h - G_\ell(u)}}.$$

Moreover, for large-amplitude regimes we have the following behaviour.

- If $\alpha \geq -1$, the periodic orbits fill the whole half-plane $\{u > 0\}$, and

$$T(h) \rightarrow \pi \quad \text{as } h \rightarrow +\infty.$$

- If $-3 < \alpha < -1$, a finite-energy separatrix appears at $h = 0$ and

$$T(h) \rightarrow \frac{2\pi}{\alpha+3} \quad \text{as } h \rightarrow 0^-.$$

For completeness, we briefly justify these limiting values of the period function. If $\alpha > -1$, set

$$A = \frac{c}{\ell^2(\alpha+1)}.$$

Let $u_-(E) < u_0 < u_+(E)$ denote the two inverse branches of G_ℓ . Splitting the period integral at u_0 and using $E = G_\ell(u)$, we obtain

$$T(h) = \sqrt{2} \int_{G_\ell(u_0)}^h \frac{u'_+(E) - u'_-(E)}{\sqrt{h - E}} dE.$$

As $E \rightarrow +\infty$,

$$u_+(E) \sim \sqrt{2E}, \quad u'_+(E) \sim \frac{1}{\sqrt{2E}},$$

whereas

$$u_-(E) \sim \left(\frac{A}{E}\right)^{1/(\alpha+1)}, \quad u'_-(E) = O(E^{-(\alpha+2)/(\alpha+1)}).$$

It follows, after the rescaling $E = hs$, that the contribution of the negative branch vanishes, while

$$\sqrt{2} \int_{G_\ell(u_0)}^h \frac{u'_+(E)}{\sqrt{h-E}} dE \rightarrow \int_0^1 \frac{ds}{\sqrt{s(1-s)}} = \pi.$$

The same conclusion holds for $\alpha = -1$, since in this case

$$G_\ell(u) = \frac{1}{2}u^2 - \frac{c}{\ell^2} \log u,$$

the positive branch still satisfies $u_+(E) \sim \sqrt{2E}$, while the negative branch tends to zero exponentially. Hence

$$T(h) \rightarrow \pi \quad \text{as } h \rightarrow +\infty.$$

Assume now that $-3 < \alpha < -1$ and set

$$k = -\frac{c}{\ell^2(\alpha+1)}.$$

Then

$$G_\ell(u) = \frac{1}{2}u^2 - ku^{-\alpha-1}.$$

The zero-energy level has turning points 0 and

$$u_s = (2k)^{1/(\alpha+3)}.$$

As $h \rightarrow 0^-$, the two turning points $u_1(h)$ and $u_2(h)$ of the periodic orbit tend to 0 and u_s , respectively. Since the limiting integrand is integrable at both endpoints, one obtains

$$\lim_{h \rightarrow 0^-} T(h) = \sqrt{2} \int_0^{u_s} \frac{du}{\sqrt{ku^{-\alpha-1} - \frac{1}{2}u^2}} = \frac{4}{\alpha+3} \int_0^1 \frac{dz}{\sqrt{1-z^2}} = \frac{2\pi}{\alpha+3},$$

where we have done the substitution $z = (u/u_s)^{(\alpha+3)/2}$.

We summarize the above information as follows, using the standard notation $(a, b) = (b, a)$ for intervals with reversed order endpoints.

Proposition 12. *If $\alpha \in (-3, -1) \setminus \{-2\}$, then the period function $T(h)$ covers the interval*

$$\left(\frac{2\pi}{\sqrt{\alpha+3}}, \frac{2\pi}{\alpha+3} \right),$$

and if $\alpha \in [-1, +\infty) \setminus \{1\}$ it covers

$$\left(\pi, \frac{2\pi}{\sqrt{\alpha+3}} \right).$$

Since any $T(h)$ commensurable with 2π corresponds to a periodic solution of system $(\tilde{S}_0)$, we have a wide choice of such solutions to which the following bifurcation result applies.

Theorem 13. *Assume $\alpha \in (-3, +\infty) \setminus \{-2, 1\}$, and consider a periodic solution $\mathbf{x}_0(t)$ of $(\tilde{S}_0)$ with angular momentum $\ell_0 > 0$. Then, the same conclusion of Theorem 9 holds.*

Proof. Notice that (ii) in Definition 4 is equivalent to (21), while (iii) holds since $\alpha > -3$. For such a choice of u_0 , ℓ_0 and α , one has

$$\mathcal{K}_{\ell_0}(u_0) = 2 \left(\frac{c}{\ell_0^2} \right)^{-\frac{2}{\alpha+3}} (\alpha-1)(\alpha+2)(\alpha+3)^2,$$

hence (iv) is verified, since $\alpha \notin \{-2, 1\}$. Moreover, $\mathcal{F}_0(\rho) = c\rho^\alpha$ is analytic. The result then follows from Theorem 9. $\square$

The case $\alpha = -3$ is highly degenerate, since all circular solutions of $(\tilde{S}_0)$ have the same angular momentum $\sqrt{c}$. In this case, equation (20) is linear, and either 0 is the only equilibrium, or every $u_0 \in \mathbb{R}$ is an equilibrium. Clearly enough, this case falls outside the scope of our study. On the other hand, in the case $\alpha < -3$, the unique equilibrium (21) is a saddle, which can be continued for small ε in a local branch of 2π -periodic solutions (see, e.g., [10, Theorem 1.1, Ch. 14]). These, in turn, provide the existence of periodic solutions of system $(\tilde{S}_\varepsilon)$ near a circular solution. We do not enter into further details, for brevity.

The resonant cases. We now deal with the cases $\alpha = 1$ and $\alpha = -2$.

If $\alpha = 1$, equation (19) becomes

$$u'' + u = \frac{c + \varepsilon b(\theta)}{\ell^2 u^3}. \quad (22)$$

In this case, we can have some resonance phenomena.

In [24] it was proved that, taking the π -periodic function defined by

$$b(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \pi/2), \\ 1, & \text{if } \theta \in [\pi/2, \pi), \end{cases}$$

then, for every $\varepsilon \neq 0$ such that $c + \varepsilon b(\theta) > 0$, all solutions of (22) are unbounded. The solutions are positive and perform increasing oscillations with the relative minima (resp. maxima) tending to zero (resp. $+\infty$). Clearly, this result reflects on our original equation (1), which can now be written as

$$\ddot{\mathbf{x}} = -(c + \varepsilon b(\theta))\mathbf{x}, \quad (23)$$

where the distance of any orbit to the origin will experience the same behaviour. However, we remark that the function $b(\theta)$ in this example is not continuous.

Let us also mention the result in [12], where it has been proved that, if $b(\theta) = \cos(2\theta)$ and $\varepsilon \neq 0$ is such that $c + \varepsilon b(\theta) > 0$, then equation (22) has no $2\pi n$ -periodic solutions, for any positive integer n . This implies that, for such a $b(\theta)$, equation (23) cannot have periodic solutions, if $\varepsilon \neq 0$ is small enough.

If $\alpha = -2$, then the original system is a kind of anisotropic Kepler problem (2). Writing $a(\theta) = \ell^{-2}(c + \varepsilon b(\theta))$, we can assume that $a(\theta)$ is a positive function, and equation (BC_ε) becomes

$$u'' + u = a(\theta). \quad (24)$$

For such an equation, linear resonance can arise; indeed, from the basic Fredholm's alternative, (24) has a 2π -periodic solution if and only if $\int_0^{2\pi} a(\theta)e^{i\theta}d\theta = 0$. In this case, all the solutions of (24) are 2π -periodic, and since $a(\theta) > 0$ for all θ , from the main result of [2] we know that at least one of them is positive. Near this one there are infinitely many positive 2π -periodic solutions, all of them corresponding to periodic trajectories of the anisotropic Kepler problem (2). If $a(\theta)$ is constant, they would be the elliptic orbits of the classical Kepler problem. Given a positive 2π -periodic solution u of (24), the corresponding solution $\mathbf{x}$ of (2) is T -periodic with

$$T = \int_0^{2\pi} \frac{ds}{\ell u(s)^2}.$$

In contrast, if the solution of (24) reaches zero, by the inversion $\rho = 1/u$ this means that the corresponding orbit $\mathbf{x}$ of (2) goes to infinity when $t \rightarrow +\infty$. For a constant $a(\theta)$, it would correspond to either a parabolic or a hyperbolic orbit of the classical Kepler problem.

If $\int_0^{2\pi} a(\theta)e^{i\theta}d\theta \neq 0$, then any solution of (24) contains a nontrivial resonant term of the form $A\theta \cos \theta + B\theta \sin \theta$, and therefore it changes sign for θ large enough. In particular, any positive branch reaches zero at some finite θ^* . Consequently, all the orbits of (2) are unbounded. More precisely, $\rho(t) = 1/u(\theta(t)) \rightarrow +\infty$ as $t \rightarrow +\infty$.

Remark 14. Equation (19) has been widely studied in the context of Convex Geometry, concretely for the planar L_p -Minkowski problem [26] and for the anisotropic shortening curve problem [9, Eq. (4.5)]. In the first case, one looks for $(2\pi/k)$ -periodic solutions, as periods higher than 2π have no geometrical meaning. See also [8, 11, 12, 21]. For the anisotropic shortening curve problem, our subharmonic solutions would be locally convex curves with self-intersections, in analogy with the Abresch-Langer curves of the isotropic case, see for instance [1, Fig.1] or [18, Figs.4 to 7].

6 Eliminating resonances

In order to try to avoid possible resonances, we consider the system

$$\ddot{\mathbf{x}} = -(c + \varepsilon b_1(\theta))|\mathbf{x}|^{\alpha-1}\mathbf{x} - (d + \varepsilon b_2(\theta))|\mathbf{x}|^{\beta-1}\mathbf{x}, \quad (\hat{S}_\varepsilon)$$

for some positive constants c and d , and $\alpha \neq \beta$. We thus have

$$\mathcal{F}_0(\rho) = c\rho^\alpha + d\rho^\beta.$$

Assume $\alpha > -3$ and $\beta > -3$, and fix $\ell_0 > 0$. Then there exists a unique $\rho_0 > 0$ such that

$$\ell_0^2 = c\rho_0^{\alpha+3} + d\rho_0^{\beta+3}.$$

Define $\lambda : (0, +\infty) \rightarrow (0, 1)$ as

$$\lambda(\rho) = \frac{c\rho^{\alpha+3}}{c\rho^{\alpha+3} + d\rho^{\beta+3}},$$

and set $u_0 = 1/\rho_0$. Then, recalling (6) and (8), we have $g_{\ell_0}(u_0) = 0$, $g'_{\ell_0}(u_0) > 0$, and

$$\mathcal{K}_{\ell_0}(u_0) = \rho_0^2 Q_{\alpha,\beta}(\lambda(\rho_0)),$$

where

$$\begin{aligned} Q_{\alpha,\beta}(\lambda) = & 2(\alpha-1)(\alpha+2)(\alpha+3)^2 \lambda^2 \\ & - (\alpha+3)(\beta+3)(3\alpha^2 - 10\alpha\beta - 2\alpha + 3\beta^2 - 2\beta + 8)\lambda(1-\lambda) \\ & + 2(\beta-1)(\beta+2)(\beta+3)^2 (1-\lambda)^2. \end{aligned}$$

Lemma 15. Assume $\alpha > -3$, $\beta > -3$, and $\alpha \neq \beta$. Then the set $\{\rho > 0 : Q_{\alpha,\beta}(\lambda(\rho)) = 0\}$ has at most two elements.

Proof. Since $\alpha \neq \beta$, the map $\rho \mapsto \lambda(\rho)$ is one-to-one from $(0, +\infty)$ onto $(0, 1)$. Therefore, the number of radii $\rho > 0$ such that $Q_{\alpha,\beta}(\lambda(\rho)) = 0$ coincides with the number of zeros of

the polynomial $Q_{\alpha,\beta}(\lambda)$ in the interval $(0, 1)$. Now, $Q_{\alpha,\beta}$ is a polynomial of degree at most two in the variable λ , so it has at most two zeros unless it is identically zero. However,

$$Q_{\alpha,\beta}(0) = 2(\beta - 1)(\beta + 2)(\beta + 3)^2,$$

and

$$Q_{\alpha,\beta}(1) = 2(\alpha - 1)(\alpha + 2)(\alpha + 3)^2,$$

hence $Q_{\alpha,\beta} \equiv 0$ would imply $\beta \in \{-2, 1\}$ and $\alpha \in \{-2, 1\}$. Since $\alpha \neq \beta$, the only possibilities are $(\alpha, \beta) = (1, -2)$ or $(\alpha, \beta) = (-2, 1)$. However, in both cases a direct computation shows that neither $Q_{1,-2}$ nor $Q_{-2,1}$ are the zero polynomial. This contradiction proves that $Q_{\alpha,\beta}$ is not identically zero. $\square$

Here is our bifurcation result.

Theorem 16. *Assume $\alpha > -3$ and $\beta > -3$, with $\alpha \neq \beta$. Consider a periodic solution $\mathbf{x}_0(t)$ of $(\widehat{S}_0)$ with angular momentum $\ell_0 > 0$. Then, for every $\sigma > 0$ and any positive integer N , there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, system $(\widehat{S}_\varepsilon)$ has at least N periodic solutions $\mathbf{x}(t)$, with angular momentum in $(\ell_0 - \sigma, \ell_0 + \sigma)$, and*

$$\text{dist}((\mathbf{x}(t), \dot{\mathbf{x}}(t)), \mathcal{M}) \leq \sigma, \quad \text{for every } t \in \mathbb{R}, \quad (25)$$

where, we recall,

$$\mathcal{M} = \{(e^{i\omega} \mathbf{x}_0(s), e^{i\omega} \dot{\mathbf{x}}_0(s)) : \omega, s \in \mathbb{R}\}.$$

Proof. For every $\ell > 0$, let $\rho_\ell > 0$ be the unique solution of

$$\ell^2 = c\rho^{\alpha+3} + d\rho^{\beta+3},$$

and set

$$u_\ell = \frac{1}{\rho_\ell}.$$

Then

$$g_\ell(u_\ell) = 0, \quad g'_\ell(u_\ell) > 0.$$

Moreover, the uniqueness of ρ_ℓ implies that

$$g_\ell(u) = 0 \iff u = u_\ell, \quad (26)$$

for every $\ell > 0$. By Lemma 15, there are at most two values of $\ell > 0$ for which $\mathcal{K}_\ell(u_\ell) = 0$. If $\mathcal{K}_{\ell_0}(u_{\ell_0}) \neq 0$, then the pair (u_{ℓ_0}, ℓ_0) is admissible, and since $\mathcal{F}_0(\rho) = c\rho^\alpha + d\rho^\beta$ is analytic on $(0, +\infty)$, Theorem 9 applies directly to $\mathbf{x}_0$, and the conclusion follows.

Assume now that $\mathcal{K}_{\ell_0}(u_{\ell_0}) = 0$. We shall construct a periodic solution $\tilde{\mathbf{x}}_0$ of $(\widehat{S}_0)$, with angular momentum $\tilde{\ell}_0$ arbitrarily close to ℓ_0 , such that

$$\mathcal{K}_{\tilde{\ell}_0}(u_{\tilde{\ell}_0}) \neq 0, \quad (27)$$

and such that the manifold associated with $\tilde{\mathbf{x}}_0$ is arbitrarily close to the manifold $\mathcal{M}$ associated with $\mathbf{x}_0$. We distinguish the circular and the non-circular cases.

The circular case. Assume first that $\mathbf{x}_0$ is circular. Since there are at most two exceptional values of ℓ , we can choose a value $\tilde{\ell}_0 > 0$, arbitrarily close to ℓ_0 , such that (27) holds. Let $\tilde{\mathbf{x}}_0$ be the circular solution of $(\hat{S}_0)$ having angular momentum $\tilde{\ell}_0$ and radius $\rho_{\tilde{\ell}_0}$, and define

$$\widetilde{\mathcal{M}} = \{ (e^{i\omega} \tilde{\mathbf{x}}_0(s), e^{i\omega} \dot{\tilde{\mathbf{x}}}_0(s)) : \omega, s \in \mathbb{R} \}.$$

Since ρ_ℓ depends continuously on ℓ , we have that $d_H(\widetilde{\mathcal{M}}, \mathcal{M}) \rightarrow 0$ as $\tilde{\ell}_0 \rightarrow \ell_0$, where d_H denotes the Hausdorff distance. Hence, we may choose $\tilde{\ell}_0$ in such a way that

$$|\tilde{\ell}_0 - \ell_0| < \frac{\sigma}{2}, \quad \text{and} \quad d_H(\widetilde{\mathcal{M}}, \mathcal{M}) < \frac{\sigma}{2}.$$

The non-circular case. Assume now that $\mathbf{x}_0$ is non-circular, and let $u_*(\theta)$ be the corresponding periodic solution of

$$u'' + g_{\ell_0}(u) = 0.$$

Denote its orbit in the Binet–Clairaut phase plane by Γ_* . After translating the angular variable, we may assume that

$$u_*(0) = q_0, \quad u'_*(0) = 0,$$

where q_0 is the maximum value of u_* on Γ_* . In particular, $q_0 > u_{\ell_0}$. Let $p_0 < u_{\ell_0}$ be the other turning point of Γ_* . Choose a primitive G_ℓ of g_ℓ . It depends analytically on (ℓ, u) for $\ell > 0$ and $u > 0$. Then $G_{\ell_0}(p_0) = G_{\ell_0}(q_0)$, and since neither turning point is an equilibrium,

$$g_{\ell_0}(p_0) \neq 0, \quad g_{\ell_0}(q_0) \neq 0.$$

By the Implicit Function Theorem, for (ℓ, q) sufficiently close to (ℓ_0, q_0) there exists a unique function $p = p(\ell, q)$, close to p_0 , such that

$$G_\ell(p(\ell, q)) = G_\ell(q).$$

The corresponding energy level contains the closed orbit

$$\Gamma_{\ell, q} = \left\{ (u, v) : \frac{1}{2}v^2 + G_\ell(u) = G_\ell(q), \quad p(\ell, q) \leq u \leq q \right\}.$$

Moreover,

$$d_H(\Gamma_{\ell, q}, \Gamma_*) \rightarrow 0 \quad \text{as} \quad (\ell, q) \rightarrow (\ell_0, q_0). \quad (28)$$

Indeed, the orbit can be written as

$$v = \pm \sqrt{2(G_\ell(q) - G_\ell(u))}, \quad p(\ell, q) \leq u \leq q,$$

and the assertion follows from the continuous dependence of $p(\ell, q)$ and G_ℓ on the parameters.

We now choose a non-exceptional value $\tilde{\ell}_0$ arbitrarily close to ℓ_0 . For this fixed value of $\tilde{\ell}_0$, let $P_{\tilde{\ell}_0}(q)$ denote the minimal period of the orbit $\Gamma_{\tilde{\ell}_0, q}$, for q in a neighborhood of q_0 . We claim that $P_{\tilde{\ell}_0}$ is not constant on any nontrivial interval. Indeed, by Lemma 10, the period function is analytic on the period annulus surrounding $(u_{\tilde{\ell}_0}, 0)$. On the other hand, since (27) holds, Lemma 5 implies that the period function is not constant near the center. By analyticity, it therefore cannot be constant on any nontrivial subinterval of the period annulus.

Consequently, in every neighborhood I of q_0 the set $P_{\tilde{\ell}_0}(I)$ is a nontrivial interval. Since $2\pi\mathbb{Q}$ is dense in $(0, +\infty)$, we can choose $\tilde{q}_0$ arbitrarily close to q_0 such that

$$P_{\tilde{\ell}_0}(\tilde{q}_0) \in 2\pi\mathbb{Q}. \quad (29)$$

Set $\tilde{\Gamma} = \Gamma_{\tilde{\ell}_0, \tilde{q}_0}$. By (29) and Proposition 3, $\tilde{\Gamma}$ corresponds to a periodic solution $\tilde{\mathbf{x}}_0$ of $(\hat{S}_0)$ having angular momentum $\tilde{\ell}_0 > 0$. We write

$$\tilde{\mathbf{x}}_0(s) = \tilde{\rho}_0(s)e^{i\tilde{\theta}_0(s)},$$

and let $\tilde{u}_*(\theta)$ be the corresponding solution of the Binet–Clairaut equation.

Defining $\tilde{\mathcal{M}}$ as above, we now show that

$$d_H(\tilde{\mathcal{M}}, \mathcal{M}) \rightarrow 0 \quad \text{as } (\tilde{\ell}_0, \tilde{q}_0) \rightarrow (\ell_0, q_0). \quad (30)$$

Indeed, let

$$(e^{i\omega}\tilde{\mathbf{x}}_0(s), e^{i\omega}\dot{\tilde{\mathbf{x}}}_0(s)) \in \tilde{\mathcal{M}}$$

and set

$$\varphi = \tilde{\theta}_0(s).$$

Then $(\tilde{u}_*(\varphi), \tilde{u}'_*(\varphi)) \in \tilde{\Gamma}$, and we can choose $\psi \in \mathbb{R}$ such that

$$|(\tilde{u}_*(\varphi), \tilde{u}'_*(\varphi)) - (u_*(\psi), u'_*(\psi))| \leq d_H(\tilde{\Gamma}, \Gamma_*),$$

as $(\tilde{\ell}_0, \tilde{q}_0) \rightarrow (\ell_0, q_0)$. Since θ_0 is strictly increasing, we can choose $r \in \mathbb{R}$ such that $\theta_0(r) = \psi$. Set $\omega_* = \omega + \varphi - \psi$. Using Proposition 1, we have

$$e^{i\omega}\tilde{\mathbf{x}}_0(s) = \frac{e^{i(\omega+\varphi)}}{\tilde{u}_*(\varphi)},$$

whereas

$$e^{i\omega_*}\mathbf{x}_0(r) = \frac{e^{i(\omega_*+\psi)}}{u_*(\psi)} = \frac{e^{i(\omega+\varphi)}}{u_*(\psi)}.$$

Hence,

$$\left| e^{i\omega}\tilde{\mathbf{x}}_0(s) - e^{i\omega_*}\mathbf{x}_0(r) \right| = \left| \frac{1}{\tilde{u}_*(\varphi)} - \frac{1}{u_*(\psi)} \right|.$$

Since the two Binet–Clairaut orbits remain in a fixed compact subset of $(0, +\infty) \times \mathbb{R}$, the right-hand side tends uniformly to zero as $d_H(\tilde{\Gamma}, \Gamma_*) \rightarrow 0$. Moreover, by (4),

$$e^{i\omega} \dot{\tilde{\mathbf{x}}}_0(s) = \tilde{\ell}_0 [-\tilde{u}'_*(\varphi) + i\tilde{u}_*(\varphi)] e^{i(\omega+\varphi)},$$

while

$$e^{i\omega_*} \dot{\mathbf{x}}_0(r) = \ell_0 [-u'_*(\psi) + iu_*(\psi)] e^{i(\omega+\varphi)}.$$

Thus

$$\begin{aligned} & \left| e^{i\omega} \dot{\tilde{\mathbf{x}}}_0(s) - e^{i\omega_*} \dot{\mathbf{x}}_0(r) \right| \\ & \leq |\tilde{\ell}_0 - \ell_0| \sqrt{\tilde{u}_*(\varphi)^2 + \tilde{u}'_*(\varphi)^2} \\ & \quad + \ell_0 \sqrt{|\tilde{u}_*(\varphi) - u_*(\psi)|^2 + |\tilde{u}'_*(\varphi) - u'_*(\psi)|^2}. \end{aligned}$$

The right-hand side tends uniformly to zero as $d_H(\tilde{\Gamma}, \Gamma_*) \rightarrow 0$.

Interchanging the roles of Γ_* and $\tilde{\Gamma}$ gives the reverse estimate. Therefore, recalling (28), we have proved (30).

By choosing first $\tilde{\ell}_0$ sufficiently close to ℓ_0 and then $\tilde{q}_0$ sufficiently close to q_0 , we may thus assume that

$$|\tilde{\ell}_0 - \ell_0| < \frac{\sigma}{2}, \quad \text{and} \quad d_H(\tilde{\mathcal{M}}, \mathcal{M}) < \frac{\sigma}{2}.$$

In either the circular or the non-circular case, we have constructed a periodic solution $\tilde{\mathbf{x}}_0$ of $(\tilde{S}_0)$, with angular momentum $\tilde{\ell}_0$, such that

$$\mathcal{K}_{\tilde{\ell}_0}(u_{\tilde{\ell}_0}) \neq 0, \quad |\tilde{\ell}_0 - \ell_0| < \frac{\sigma}{2}, \quad \text{and} \quad d_H(\tilde{\mathcal{M}}, \mathcal{M}) < \frac{\sigma}{2}.$$

The pair $(u_{\tilde{\ell}_0}, \tilde{\ell}_0)$ is admissible, and condition (26) holds for $\tilde{\ell}_0$. Therefore Theorem 9, applied to $\tilde{\mathbf{x}}_0$ with $\sigma/2$ in place of σ , yields, for $|\varepsilon|$ sufficiently small, at least N periodic solutions $\mathbf{x}(t)$ of $(\hat{S}_\varepsilon)$, all having angular momentum $\tilde{\ell}_0$, and satisfying

$$\text{dist}\left((\mathbf{x}(t), \dot{\mathbf{x}}(t)), \tilde{\mathcal{M}}\right) \leq \frac{\sigma}{2}, \quad \text{for every } t \in \mathbb{R}.$$

Since

$$|\tilde{\ell}_0 - \ell_0| < \frac{\sigma}{2},$$

their angular momentum belongs to $(\ell_0 - \sigma, \ell_0 + \sigma)$. Finally,

$$\text{dist}\left((\mathbf{x}(t), \dot{\mathbf{x}}(t)), \mathcal{M}\right) \leq \text{dist}\left((\mathbf{x}(t), \dot{\mathbf{x}}(t)), \tilde{\mathcal{M}}\right) + d_H(\tilde{\mathcal{M}}, \mathcal{M}) \leq \sigma,$$

for every $t \in \mathbb{R}$. This proves (25), thus completing the proof. $\square$

A particularly interesting situation occurs when

$$\mathcal{F}_0(\rho) = c\rho^{-2} + d\rho.$$

This combines the two exceptional exponents $\alpha = -2$ and $\beta = 1$, for which the quantity $\mathcal{K}_{\ell_0}(u_0)$ vanishes in the single-power case. In this case, we have

$$Q_{-2,1}(\lambda) = -180\lambda(1-\lambda).$$

Recalling that $\lambda_0 \in (0, 1)$, we have that $\mathcal{K}_{\ell_0}(u_0) = \rho_0^2 Q_{-2,1}(\lambda_0) < 0$, hence the pair (u_0, ℓ_0) is admissible, and in Theorem 16 we will find all our periodic solutions with the same angular momentum ℓ_0 .

A Appendix: Proofs of Lemmas 5 and 10

In this section, for the reader's convenience, we provide the detailed proofs of Lemmas 5 and 10.

Proof of Lemma 5. We apply the classical Lindstedt–Poincaré technique (see, e.g., [23, §7.3], or [15]). Let (u_0, ℓ_0) be admissible. We consider the solutions u of (9) such that $u(0) = u_0 + a$ and $u'(0) = 0$, where a is a small parameter. Our aim is to provide an estimate of the period function $\tau(a)$.

Writing $u(\theta) = u_0 + x(\theta)$, from (9) we have that

$$x''(\theta) + g_{\ell_0}(u_0 + x(\theta)) = 0, \quad x(0) = a, \quad x'(0) = 0.$$

We expand g_{ℓ_0} near u_0 and, recalling that $g_{\ell_0}(u_0) = 0$ and $g'_{\ell_0}(u_0) > 0$, we write the equation for x as

$$x'' + \omega_0^2 x + c_2 x^2 + c_3 x^3 = o(x^3), \tag{31}$$

where $\omega_0^2 = g'_{\ell_0}(u_0)$, $c_2 = \frac{1}{2}g''_{\ell_0}(u_0)$, and $c_3 = \frac{1}{6}g'''_{\ell_0}(u_0)$.

Let us denote by $\omega(a)$ the frequency of x :

$$\omega(a) = \frac{2\pi}{\tau(a)}.$$

It is convenient to introduce the rescaling $s = \omega(a)\theta$, and to express x in the variable s . Problem (31) thus becomes

$$\omega(a)^2 x'' + \omega_0^2 x + c_2 x^2 + c_3 x^3 = o(x^3), \quad x(0) = a, \quad x'(0) = 0, \tag{32}$$

where the derivatives are now with respect to s .

We expand $\omega(a)$ near 0 as:

$$\omega(a) = \omega_0 + \nu_1 a + \nu_2 a^2 + o(a^2), \quad (33)$$

and from (32) we get

$$(\omega_0 + \nu_1 a + \nu_2 a^2 + o(a^2))^2 x'' + \omega_0^2 x + c_2 x^2 + c_3 x^3 = o(x^3).$$

Since $(\omega_0 + \nu_1 a + \nu_2 a^2 + o(a^2))^2 = \omega_0^2 + 2\omega_0 \nu_1 a + (\nu_1^2 + 2\omega_0 \nu_2) a^2 + o(a^2)$, we obtain

$$\omega_0^2(x'' + x) + (2\omega_0 \nu_1 a + (\nu_1^2 + 2\omega_0 \nu_2) a^2 + o(a^2)) x'' + c_2 x^2 + c_3 x^3 = o(x^3). \quad (34)$$

This suggests to expand the solution x as

$$x(s) = a \cos s + a^2 x_2(s) + a^3 x_3(s) + o(a^3). \quad (35)$$

From the initial conditions for x we have $x_2(0) = x_2'(0) = 0$ and $x_3(0) = x_3'(0) = 0$. Substituting (35) in (34), after rearranging the terms in powers of a , we can write

$$\begin{aligned} a^2(\omega_0^2 x_2'' - 2\omega_0 \nu_1 \cos s + \omega_0^2 x_2 + c_2 \cos^2 s) \\ + a^3(\omega_0^2 x_3'' + 2\omega_0 \nu_1 x_2'' - (2\nu_2 \omega_0 + \nu_1^2) \cos s + \omega_0^2 x_3 + 2c_2 x_2 \cos s + c_3 \cos^3 s) \\ + o(a^3) = 0. \end{aligned}$$

At the second order we get

$$\omega_0^2(x_2'' + x_2) = 2\omega_0 \nu_1 \cos s - c_2 \cos^2 s.$$

Since the solution $x(s)$ is 2π -periodic, the first term in the right-hand side is resonant, so it must be equal to zero (i.e., $\nu_1 = 0$). Hence, the above reduces to

$$x_2'' + x_2 = -\frac{c_2}{2\omega_0^2} (1 + \cos(2s)), \quad x_2(0) = 0 \quad x_2'(0) = 0,$$

so that

$$x_2(s) = -\frac{c_2}{2\omega_0^2} + \frac{c_2}{6\omega_0^2} \cos(2s) + \frac{c_2}{3\omega_0^2} \cos s.$$

At the third order, recalling that $\nu_1 = 0$, we obtain

$$x_3'' + x_3 = \frac{2\nu_2}{\omega_0} \cos s + \frac{c_2^2}{\omega_0^4} \cos s \left(1 - \frac{1}{3} \cos(2s) - \frac{2}{3} \cos s\right) - \frac{c_3}{\omega_0^2} \cos^3 s.$$

Basic trigonometric formulas give

$$x_3'' + x_3 = \left[\frac{2\nu_2}{\omega_0} + \frac{5}{6} \frac{c_2^2}{\omega_0^4} - \frac{3}{4} \frac{c_3}{\omega_0^2} \right] \cos s - \frac{c_2^2}{3\omega_0^4} (1 + \cos(2s)) + \left[-\frac{1}{6} \frac{c_2^2}{\omega_0^4} - \frac{1}{4} \frac{c_3}{\omega_0^2} \right] \cos(3s).$$

Again, the first term in the right-hand side is resonant and must be canceled out, so that

$$\nu_2 = -\frac{5c_2^2}{12\omega_0^3} + \frac{3c_3}{8\omega_0} = -\frac{5[g_{\ell_0}''(u_0)]^2}{48\omega_0^3} + \frac{g_{\ell_0}'''(u_0)}{16\omega_0} = -\frac{\mathcal{K}_{\ell_0}(u_0)}{48\omega_0^3}.$$

Substituting this and $\nu_1 = 0$ into (33), by the definition of c_2 and c_3 we have

$$\omega(a) = \omega_0 \left[1 - \frac{\mathcal{K}_{\ell_0}(u_0)}{48\omega_0^4} a^2 + o(a^2) \right].$$

Hence, we finally obtain, using the approximation $(1 + \epsilon)^{-1} \sim (1 - \epsilon)$ for small ϵ ,

$$\tau(a) = \frac{2\pi}{\omega(a)} = \frac{2\pi}{\omega_0} \left[1 + \frac{\mathcal{K}_{\ell_0}(u_0)}{48\omega_0^4} a^2 + o(a^2) \right],$$

and the proof is complete. $\square$

Proof of Lemma 10. We prove that T is analytic on $\mathcal{J}_0 = (0, \beta_0)$. For each $h \in \mathcal{J}_0$, let $u_1(h) < 0 < u_2(h)$ be the turning points, characterized by

$$V(u_1(h)) = V(u_2(h)) = h.$$

The period of the corresponding periodic orbit is

$$T(h) = \sqrt{2} \int_{u_1(h)}^{u_2(h)} \frac{du}{\sqrt{h - V(u)}}. \quad (36)$$

Fix $h_0 \in \mathcal{J}_0$. Since $u_1(h_0)$ and $u_2(h_0)$ are distinct from the center, they are simple zeros of $V(u) - h_0$, hence

$$V'(u_1(h_0)) \neq 0, \quad V'(u_2(h_0)) \neq 0.$$

By the Implicit Function Theorem, after possibly shrinking to a neighborhood I of h_0 , the functions $u_1(h)$ and $u_2(h)$ are analytic on I . Set

$$\Phi(u, h) := h - V(u).$$

For each $h \in I$, the function $u \mapsto \Phi(u, h)$ vanishes simply at $u = u_1(h)$ and $u = u_2(h)$. Therefore,

$$\Psi(u, h) := \frac{\Phi(u, h)}{(u - u_1(h))(u_2(h) - u)}$$

extends analytically across the apparent singularities at $u = u_1(h)$ and $u = u_2(h)$; equivalently, Ψ is analytic in a neighborhood of the compact set

$$K := \{(u, h) : h \in \bar{I}, u_1(h) \leq u \leq u_2(h)\},$$

where $\bar{I}$ denotes the closure of I . Moreover, for $h \in I$ and $u \in (u_1(h), u_2(h))$,

$$\Phi(u, h) > 0, \quad (u - u_1(h))(u_2(h) - u) > 0,$$

hence $\Psi(u, h) > 0$. At the endpoints, using the analytic extension,

$$\Psi(u_1(h), h) = \lim_{u \rightarrow u_1(h)} \frac{\Phi(u, h)}{(u - u_1(h))(u_2(h) - u)} = \frac{\partial_u \Phi(u_1(h), h)}{u_2(h) - u_1(h)} = \frac{-V'(u_1(h))}{u_2(h) - u_1(h)},$$

and

$$\Psi(u_2(h), h) = \lim_{u \rightarrow u_2(h)} \frac{\Phi(u, h)}{(u - u_1(h))(u_2(h) - u)} = \frac{V'(u_2(h))}{u_2(h) - u_1(h)}.$$

Since $u_1(h)$ and $u_2(h)$ are the left and right turning points, respectively, one has

$$V'(u_1(h)) < 0, \quad V'(u_2(h)) > 0,$$

so both endpoint values are positive. Thus $\Psi > 0$ on the compact set K . By continuity, after possibly shrinking the neighborhood of K , we may assume that $\Psi(u, h) > 0$ throughout that neighborhood. Now, substituting

$$h - V(u) = (u - u_1(h))(u_2(h) - u)\Psi(u, h)$$

into (36) and performing the change of variables

$$u = u_1(h) + (u_2(h) - u_1(h)) \sin^2 \theta, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

we obtain

$$T(h) = 2\sqrt{2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{\Psi(u_1(h) + (u_2(h) - u_1(h)) \sin^2 \theta, h)}}.$$

The integrand is analytic in (θ, h) in a neighborhood of $[0, \pi/2] \times \bar{I}$, hence T is analytic on I . Since h_0 was arbitrary, T is analytic on $\mathcal{J}_0$. $\square$

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