

# An extension of Bertrand's Theorem

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## Abstract

We extend the celebrated Bertrand Theorem to planar central-force systems driven by the  $p$ -Laplacian operator. The case  $p = 2$  is well known, and different proofs already exist. We prove that, requiring boundedness of trajectories starting near all circular orbits and periodicity of every bounded motion, it necessarily has to be that  $p = 2$ , and hence the potential must be one of the two classical ones: the Kepler potential or the isotropic harmonic potential.

## 1 Introduction

In 1873, Bertrand [2] proved the following remarkable phenomenon for planar motions under central forces: among the Newtonian systems

$$\ddot{\mathbf{x}} + V'(|\mathbf{x}|) \frac{\mathbf{x}}{|\mathbf{x}|} = 0, \quad \mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^2,$$

the requirement that *all bounded motions with nonzero angular momentum are periodic* implies that  $V$  has to be either the Kepler potential  $V(r) = -k/r$  or the isotropic harmonic potential  $V(r) = kr^2$ . Bertrand's result has since become a cornerstone of classical mechanics and celestial dynamics, and it is often regarded as a paradigmatic example of how global geometric properties of trajectories constrain the underlying interaction law. Different proofs have been proposed (see, e.g., [1, 3, 4, 5, 6, 9, 14, 15]).

A delicate point, already implicit in Bertrand's discussion and emphasized in later literature, is that the slogan "*all bounded orbits are closed*" must be interpreted with care. Indeed, taken literally as a bare logical statement, it may hold for *vacuous* reasons: a potential may admit no bounded trajectories at all, or it may admit bounded trajectories only in a family of circular motions. Such cases do not reflect the intended content of Bertrand's theorem, hence the assumption will need to be more precisely stated.

The purpose of this paper is to investigate Bertrand-type phenomena for the  $p$ -Laplacian system

$$\frac{d}{dt}(|\dot{\mathbf{x}}|^{p-2} \dot{\mathbf{x}}) + V'(|\mathbf{x}|) \frac{\mathbf{x}}{|\mathbf{x}|} = 0, \quad p > 1. \quad (1)$$

Our goal is to show that a *robust* Bertrand property holds only if  $p = 2$  and either  $V(r) = -k/r$  or  $V(r) = kr^2$ . To be more precise, we need the following.

**Definition 1.1.** We say that  $V$  satisfies the B-property if the following three conditions hold:

- (i) for every  $r_0 > 0$  there exists a circular solution with radius  $r_0$ ;
- (ii) for every circular solution  $\mathbf{x}_0$  there exists a  $\delta > 0$  such that, if  $\mathbf{x}$  is any other solution satisfying

$$|\mathbf{x}(t_0) - \mathbf{x}_0(t_0)| + |\dot{\mathbf{x}}(t_0) - \dot{\mathbf{x}}_0(t_0)| < \delta,$$

for some  $t_0 \in \mathbb{R}$ , then  $\mathbf{x}$  is bounded;

- (iii) every bounded solution is periodic.

Condition (ii) is included in order to rule out the vacuous situations we were mentioning above, in which the only bounded motions are circular.

Our result shows that the B-property selects the Newtonian case  $p = 2$  within the family of  $p$ -Laplacian dynamics. Here is the statement.

**Theorem 1.2.** *If the B-property holds for equation (1), then necessarily  $p = 2$ . In such a case, either  $V(r) = -k/r$ , or  $V(r) = kr^2$ , for some positive constant  $k$ .*

Quoting Bertrand [2], “*Toutes les lois d’attraction permettent des orbites fermées, mais la loi de la nature est la seule qui les impose*” (in English [13]: “All laws of attraction allow closed orbits, but only the law of Nature imposes them”). Echoing this aphorism, one could view our  $p$ -Laplacian analysis as a selection principle for Newton’s second law of motion. This is somewhat confirmed by a recent result of Ortega and Rojas [10], who provided a computer-assisted proof of the fact that, if the differential operator is replaced by a relativistic-type operator, then there cannot be any analytical potential  $V$  for which the Bertrand property holds (see [10] for a precise statement).

The paper is organized as follows. In Section 2 we provide some preliminaries in a very general setting, with a differential operator of the type  $\frac{d}{dt}\phi(\dot{\mathbf{x}})$ . Besides permitting a comparison with the relativistic operator, we believe that presenting our considerations in this general setting will be useful for further developments in this field. Then, in Section 3 we specialize to the case  $\phi(\mathbf{v}) = |\mathbf{v}|^{p-2}\mathbf{v}$ , the one leading to equation (1), and provide the proof of Theorem 1.2, which follows the lines proposed by Arnold in [1], with a further need of investigation by a Lindstedt–Poincaré small-amplitude expansion near circular orbits. This technique goes back to Lindstedt [7] and was developed by Poincaré in connection with periodic motions in Celestial Mechanics (see [11] and the systematic treatment in [12]). In Appendix A we provide more details on how we apply this method, and in Appendix B we prove some technical lemmas.

## 2 Some preliminaries

We start by considering the general differential equation

$$\frac{d}{dt}\phi(\dot{\mathbf{x}}) + V'(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} = 0, \quad (2)$$

where

$$\phi(\mathbf{v}) = \varphi(|\mathbf{v}|)\frac{\mathbf{v}}{|\mathbf{v}|}.$$

Here,  $\varphi : ]0, \omega[ \rightarrow ]0, +\infty[$  is an increasing homeomorphism, with  $\omega \leq +\infty$ , such that  $\varphi(0) = 0$ , and it is continuously differentiable on  $]0, \omega[$  with  $\varphi'(v) > 0$  for every  $v > 0$ , and  $V : ]0, +\infty[ \rightarrow \mathbb{R}$  is a twice continuously differentiable function. We will only consider solutions  $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^2$  such that

$$\mathbf{x}(t) \neq 0 \quad \text{and} \quad \dot{\mathbf{x}}(t) \neq 0, \quad \text{for every } t \in \mathbb{R}. \quad (3)$$

**Lemma 2.1.** *Condition (i) in Definition 1.1 is equivalent to*

$$V'(r) > 0, \quad \text{for every } r > 0. \quad (4)$$

*Proof.* Fix  $r > 0$ . A circular motion of radius  $r$  has the form

$$\mathbf{x}(t) = re^{i(\Omega t + \theta_0)}, \quad \Omega \neq 0,$$

hence  $\dot{\mathbf{x}}(t) = r\Omega ie^{i(\Omega t + \theta_0)}$  and  $|\dot{\mathbf{x}}| = r|\Omega|$ . Writing  $e_r = e^{i(\Omega t + \theta_0)}$  and  $e_\theta = ie^{i(\Omega t + \theta_0)}$ , we have  $\dot{\mathbf{x}} = r\Omega e_\theta$  and therefore

$$\phi(\dot{\mathbf{x}}) = \varphi(|\dot{\mathbf{x}}|)\frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} = \varphi(r|\Omega|)\text{sgn}(\Omega)e_\theta.$$

Since  $e'_\theta = -\Omega e_r$ , we obtain

$$\frac{d}{dt}\phi(\dot{\mathbf{x}}) = \varphi(r|\Omega|)\text{sgn}(\Omega)e'_\theta = -\varphi(r|\Omega|)|\Omega|e_r.$$

Substituting into (2) yields

$$V'(r) = \varphi(r|\Omega|)|\Omega|. \quad (5)$$

In particular, if a non-stationary circular orbit exists then  $|\Omega| > 0$  and  $\varphi(r|\Omega|) > 0$ , hence  $V'(r) > 0$ .

Conversely, assume  $V'(r) > 0$  and consider the function  $F : ]0, \omega/r[ \rightarrow \mathbb{R}$  defined as  $F(s) := \varphi(rs)s$ . Since  $\varphi$  is increasing with  $\varphi(0) = 0$ , the map  $F$  is continuous, strictly increasing, satisfies  $F(0^+) = 0$ , and diverges to  $+\infty$  as  $s \rightarrow (\omega/r)^-$  (or as  $s \rightarrow +\infty$  if  $\omega = +\infty$ ). Hence there exists a unique  $s_* > 0$  such that  $F(s_*) = V'(r)$ . Setting  $\Omega := s_*$ , equation (5) holds, and the corresponding  $\mathbf{x}(t) = re^{i(\Omega t + \theta_0)}$  is a circular solution of radius  $r$ .  $\square$

$\varphi(s)$	$\frac{s}{\sqrt{1-s^2}}$	$s^{p-1}$
$\varphi^{-1}(s)$	$\frac{s}{\sqrt{1+s^2}}$	$s^{q-1}$
$\gamma(s) = \frac{\varphi(\sqrt{s})}{\sqrt{s}}$	$\frac{1}{\sqrt{1-s}}$	$s^{\frac{p-2}{2}}$
$\Psi(s) = \int \varphi^{-1}(s) ds$	$\sqrt{1+s^2}$	$\frac{1}{q} s^q$
$\Xi(s) = \varphi(s)s$	$\frac{s^2}{\sqrt{1-s^2}}$	$s^p$
$\Upsilon(s) = \varphi(\Xi^{-1}(s))$	$\sqrt{\frac{s\sqrt{s^2+4}+s^2}{2}}$	$s^{1/q}$
$\Sigma(s) = \frac{s}{\varphi^{-1}(s)}$	$\sqrt{1+s^2}$	$s^{2-q}$

Table 1: A table of functions we use in this paper and their value in the relativistic case and in the  $p$ -Laplace case. Here  $q > 1$  is such that  $(1/p) + (1/q) = 1$ .

## 2.1 Angular momentum and energy

It is useful to introduce the function  $\gamma : ]0, \omega^2[ \rightarrow ]0, +\infty[$ , defined by

$$\gamma(s) = \frac{\varphi(\sqrt{s})}{\sqrt{s}}.$$

Taking into account (3), equation (2) can thus be rewritten as

$$\frac{d}{dt} \left( \gamma(|\dot{\mathbf{x}}|^2) \dot{\mathbf{x}} \right) + V'(|\mathbf{x}|) \frac{\mathbf{x}}{|\mathbf{x}|} = 0. \quad (6)$$

Let us write a solution in polar coordinates, i.e.,  $\mathbf{x}(t) = r(t)e^{i\theta(t)}$ . Then,

$$\dot{\mathbf{x}} = (\dot{r} + ir\dot{\theta})e^{i\theta}, \quad |\dot{\mathbf{x}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2. \quad (7)$$

**Proposition 2.2.** *The pair  $(r, \theta)$  solves the system*

$$\begin{cases} \frac{d}{dt} \left( \dot{r} \gamma(\dot{r}^2 + r^2 \dot{\theta}^2) \right) - r \dot{\theta}^2 \gamma(\dot{r}^2 + r^2 \dot{\theta}^2) + V'(r) = 0, \\ \frac{d}{dt} \left( r^2 \gamma(\dot{r}^2 + r^2 \dot{\theta}^2) \dot{\theta} \right) = 0. \end{cases} \quad (8)$$

*Proof.* Set

$$\lambda(t) := \gamma(|\dot{\mathbf{x}}(t)|^2) = \gamma(\dot{r}^2(t) + r^2(t)\dot{\theta}(t)^2), \quad \mathbf{y}(t) := \lambda(t)\dot{\mathbf{x}}(t).$$

Then

$$\mathbf{y} = (\dot{r} + ir\dot{\theta})\lambda e^{i\theta}.$$

Differentiating,

$$\dot{\mathbf{y}} = \left( \left( \frac{d}{dt}(\dot{r}\lambda) - r\dot{\theta}^2\lambda \right) + i\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}\lambda) \right) e^{i\theta}.$$

So, equation (6) becomes

$$\left( \frac{d}{dt}(\dot{r}\lambda) - r\dot{\theta}^2\lambda \right) + i\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}\lambda) + V'(r) = 0.$$

Taking real and imaginary parts yields

$$\frac{d}{dt}(\dot{r}\lambda) - r\dot{\theta}^2\lambda + V'(r) = 0, \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}\lambda) = 0,$$

which gives exactly (8). □

We define the *angular momentum*

$$\ell = r^2\gamma(\dot{r}^2 + r^2\dot{\theta}^2)\dot{\theta}, \tag{9}$$

and the *energy*

$$E = \Psi(\varphi(|\dot{\mathbf{x}}|)) + V(|\mathbf{x}|), \tag{10}$$

where  $\Psi$  is a primitive of  $\varphi^{-1}$ ; the additive constant is irrelevant.

**Proposition 2.3.** *Both the angular momentum and the energy are constant along the orbit of the solution.*

*Proof.* That the angular momentum is preserved follows directly from the second equation in (8). For completeness, we also give a coordinate-free verification directly from (6). Define the scalar angular momentum as the planar cross product

$$\ell(t) := \mathbf{x}(t) \times \left( \gamma(|\dot{\mathbf{x}}(t)|^2)\dot{\mathbf{x}}(t) \right), \quad (a_1, a_2) \times (b_1, b_2) := a_1b_2 - a_2b_1.$$

In polar coordinates  $\mathbf{x} = re_r$  and  $\dot{\mathbf{x}} = \dot{r}e_r + r\dot{\theta}e_\theta$ , hence  $\ell = r^2\gamma(\dot{r}^2 + r^2\dot{\theta}^2)\dot{\theta}$ , in agreement with (9). Differentiating and using (6),

$$\dot{\ell} = \dot{\mathbf{x}} \times \left( \gamma(|\dot{\mathbf{x}}|^2)\dot{\mathbf{x}} \right) + \mathbf{x} \times \frac{d}{dt} \left( \gamma(|\dot{\mathbf{x}}|^2)\dot{\mathbf{x}} \right) = \mathbf{x} \times \frac{d}{dt} \left( \dot{\mathbf{x}} \gamma(|\dot{\mathbf{x}}|^2) \right) = -\mathbf{x} \times V'(|\mathbf{x}|) \frac{\mathbf{x}}{|\mathbf{x}|} = 0,$$

and the angular momentum is conserved.

Concerning the energy, set

$$v(t) := |\dot{\mathbf{x}}(t)|, \quad u(t) := \varphi(v(t)),$$

and recall that  $\Psi$  is a primitive of  $\varphi^{-1}$ , i.e.,  $\Psi' = \varphi^{-1}$ . Then,

$$\frac{d}{dt}\Psi(\varphi(|\dot{\mathbf{x}}|)) = \Psi'(u) u' = \varphi^{-1}(u) \frac{d}{dt}\varphi(v) = v \frac{d}{dt}\varphi(v). \quad (11)$$

Moreover, since

$$\phi(\dot{\mathbf{x}}) = \varphi(|\dot{\mathbf{x}}|) \frac{\dot{\mathbf{x}}}{|\dot{\mathbf{x}}|} = \alpha(v) \dot{\mathbf{x}}, \quad \alpha(v) := \frac{\varphi(v)}{v},$$

we have

$$\frac{d}{dt}\phi(\dot{\mathbf{x}}) = \alpha'(v) \dot{v} \dot{\mathbf{x}} + \alpha(v) \ddot{\mathbf{x}},$$

and taking the scalar product with  $\dot{\mathbf{x}}$  yields

$$\left\langle \frac{d}{dt}\phi(\dot{\mathbf{x}}), \dot{\mathbf{x}} \right\rangle = \alpha'(v) \dot{v} |\dot{\mathbf{x}}|^2 + \alpha(v) \langle \ddot{\mathbf{x}}, \dot{\mathbf{x}} \rangle.$$

Since  $\dot{v} = \langle \ddot{\mathbf{x}}, \dot{\mathbf{x}} \rangle / v$  and

$$\alpha'(v) = \frac{\varphi'(v)v - \varphi(v)}{v^2},$$

a direct computation gives

$$\left\langle \frac{d}{dt}\phi(\dot{\mathbf{x}}), \dot{\mathbf{x}} \right\rangle = \varphi'(v) \langle \ddot{\mathbf{x}}, \dot{\mathbf{x}} \rangle = v \varphi'(v) \dot{v} = v \frac{d}{dt}\varphi(v). \quad (12)$$

Combining (11) and (12), we obtain

$$\frac{d}{dt}\Psi(\varphi(|\dot{\mathbf{x}}|)) = \left\langle \frac{d}{dt}\phi(\dot{\mathbf{x}}), \dot{\mathbf{x}} \right\rangle = -V'(|\mathbf{x}|) \left\langle \frac{\mathbf{x}}{|\mathbf{x}|}, \dot{\mathbf{x}} \right\rangle = -\frac{d}{dt}V(|\mathbf{x}|).$$

We conclude that

$$\frac{d}{dt}\left(\Psi(\varphi(|\dot{\mathbf{x}}|)) + V(|\mathbf{x}|)\right) = 0,$$

proving that the energy is constant along the orbit.  $\square$

Let us summarize the above.

**Proposition 2.4.** *Let  $\mathbf{x}(t) = r(t)e^{i\theta(t)}$  satisfy (3). Then,  $\mathbf{x}$  is a solution of (6) if and only if there exists a constant  $\ell \neq 0$  such that  $(r, \theta)$  is a solution of*

$$\begin{cases} \frac{d}{dt}\left(\dot{r}\gamma(\dot{r}^2 + r^2\dot{\theta}^2)\right) - \frac{\ell^2}{r^3\gamma(\dot{r}^2 + r^2\dot{\theta}^2)} + V'(r) = 0, \\ \ell = r^2\gamma(\dot{r}^2 + r^2\dot{\theta}^2)\dot{\theta}. \end{cases} \quad (13)$$

From now on, without loss of generality, we will always consider our solutions to have a positive angular momentum, i.e.,  $\ell > 0$ . Equivalently, we assume that  $\dot{\theta}(t) > 0$  for every  $t$ .

## 2.2 Circular solutions

We now consider a circular solution  $\mathbf{x}(t) = re^{i\theta(t)}$  of (2). It is useful to introduce the invertible function  $\Xi : [0, \omega[ \rightarrow [0, +\infty[$ ,

$$\Xi(s) = \varphi(s)s,$$

and the function  $\Upsilon : [0, +\infty[ \rightarrow [0, +\infty[$ ,

$$\Upsilon(s) = \varphi(\Xi^{-1}(s)).$$

**Proposition 2.5.** *Assuming (4), for every  $r > 0$  a circular solution of (2) has angular momentum*

$$L(r) = \Upsilon(V'(r)r)r, \quad (14)$$

*it has energy*

$$E(r) = \Psi(\Upsilon(V'(r)r)) + V(r), \quad (15)$$

*and angular velocity*

$$\Omega(r) = \frac{1}{r} \Xi^{-1}(V'(r)r).$$

*Hence, the circular solutions of (2) are described by*

$$\mathbf{x}(t) = re^{i\Omega(r)(t-t_0)},$$

*where  $r > 0$  and  $t_0 \in \mathbb{R}$ .*

*Proof.* Since  $\dot{r} = 0$ , the second equation in (13) becomes

$$\ell = r^2\gamma(r^2\dot{\theta}^2)\dot{\theta} = r\varphi(r\dot{\theta}), \quad (16)$$

so that, since  $\varphi$  is invertible, the angular velocity  $\dot{\theta}$  is constant and satisfies

$$\dot{\theta} = \Omega(r, \ell) := \frac{1}{r} \varphi^{-1}\left(\frac{\ell}{r}\right). \quad (17)$$

From the first equation in (13), using (16) we obtain

$$V'(r) = \frac{\ell^2}{\gamma(r^2\Omega^2(r, \ell))r^3} = \varphi(r\Omega(r, \ell))\Omega(r, \ell) = \frac{1}{r}\Xi(r\Omega(r, \ell)).$$

As a consequence,  $\Omega(r, \ell) = \Omega(r) = \frac{1}{r}\Xi^{-1}(V'(r)r)$ , and from (17) we deduce that

$$\ell = \varphi\left(\Xi^{-1}(V'(r)r)\right)r = \Upsilon(V'(r)r)r.$$

Writing the circular solution as  $\mathbf{x}(t) = re^{i\Omega(r)(t-t_0)}$  one has  $|\dot{\mathbf{x}}| = r\Omega(r)$  and

$$E(r) = \Psi(\varphi(r\Omega(r))) + V(r).$$

Noticing that  $\varphi(r\Omega(r)) = \varphi(\Xi^{-1}(V'(r)r)) = \Upsilon(V'(r)r)$ , we conclude the proof.  $\square$

The following observation will be useful.

**Remark 2.6.** *In the above setting, the function  $L(r)$  in (14) is constant if and only if there exists  $\ell > 0$  such that*

$$V'(r) = \frac{1}{r} \Xi(\varphi^{-1}(\ell/r)) = \frac{\ell}{r^2} \varphi^{-1}(\ell/r). \quad (18)$$

*In this case,  $L(r) = \ell$  for every  $r > 0$ .*

### 2.3 The Clairaut change of variable

Let us consider a solution of (13), with  $\ell > 0$ . Then  $\theta(t)$  is invertible and we denote its inverse by  $t(\theta)$ . Let us define the function  $\Sigma : ]0, +\infty[ \rightarrow ]0, +\infty[$  as

$$\Sigma(s) := \gamma((\varphi^{-1}(s))^2) = \frac{s}{\varphi^{-1}(s)}.$$

**Proposition 2.7.** *For every solution  $\mathbf{x}(t)$  of (2) with angular momentum  $\ell > 0$ , the function  $\rho : \mathbb{R} \rightarrow ]0, +\infty[$  defined as*

$$\rho(\theta) = \frac{1}{r(t(\theta))} \quad (19)$$

*solves the differential equation*

$$\frac{d^2\rho}{d\theta^2} + \rho + \frac{\Gamma(\rho, \frac{d\rho}{d\theta}; \ell)}{\ell^2} W'(\rho) = 0, \quad (20)$$

*where  $W(\rho) = V(1/\rho)$  and*

$$\Gamma(\rho, \eta; \ell) = \Sigma\left(\ell\sqrt{\eta^2 + \rho^2}\right). \quad (21)$$

*Proof.* We compute

$$\eta := \frac{d\rho}{d\theta} = -\frac{1}{r^2} \frac{dr}{dt} \frac{dt}{d\theta} = -\frac{\dot{r}}{r^2\dot{\theta}} = -\frac{\dot{r}\gamma(\dot{r}^2 + r^2\dot{\theta}^2)}{\ell}. \quad (22)$$

Differentiating again with respect to  $\theta$  and using (13) yields

$$\frac{d^2\rho}{d\theta^2} = \left(\frac{d}{dt} \frac{d\rho}{d\theta}\right) \frac{dt}{d\theta} = -\frac{1}{\ell} \left(\frac{\ell\dot{\theta}}{r} - V'(r)\right) \frac{1}{\dot{\theta}} = -\rho + \frac{V'(r)}{\ell\dot{\theta}}.$$

Since  $W(\rho) = V(1/\rho)$ , we have that  $r^2V'(r) = -W'(\rho)$ . Recalling (9), we get

$$\frac{V'(r)}{\ell\dot{\theta}} = -\frac{1}{\ell^2} \gamma(\dot{r}^2 + r^2\dot{\theta}^2) W'(\rho).$$

Moreover, from (22) and (9), we deduce

$$\left[ \varphi \left( \sqrt{\dot{r}^2 + r^2 \dot{\theta}^2} \right) \right]^2 = \left[ \gamma(\dot{r}^2 + r^2 \dot{\theta}^2) \right]^2 (\dot{r}^2 + r^2 \dot{\theta}^2) = \ell^2 (\eta^2 + \rho^2), \quad (23)$$

hence

$$\dot{r}^2 + r^2 \dot{\theta}^2 = \left[ \varphi^{-1} \left( \ell \sqrt{\eta^2 + \rho^2} \right) \right]^2.$$

Substitution in the above completes the proof.  $\square$

We now explain how to reconstruct a solution of (2) from a solution of (20).

**Proposition 2.8.** *For every solution  $\rho(\theta)$  of (20) with  $\ell > 0$ , there is a solution  $\mathbf{x}(t) = r(t)e^{i\theta(t)}$  of (2) satisfying (19).*

*Proof.* Let  $\rho = \rho(\theta)$  be a solution of (20), with  $\ell > 0$ . Fix  $t_0 \in \mathbb{R}$  and define

$$t(\theta) = t_0 + \int_0^\theta F(\varphi) d\varphi,$$

where

$$F(\varphi) := \frac{\Gamma(\rho(\varphi), \rho'(\varphi); \ell)}{\ell \rho(\varphi)^2}. \quad (24)$$

Since the integrand is positive, the map  $\theta \mapsto t(\theta)$  is strictly increasing, hence invertible. Let  $\theta = \theta(t)$  be its inverse, and set

$$r(t) := \frac{1}{\rho(\theta(t))}.$$

Writing  $\rho = \rho(\theta(t))$ ,  $\eta = \rho'(\theta(t))$ , and  $\Gamma = \Gamma(\rho, \eta; \ell)$ , from

$$\frac{dt}{d\theta} = \frac{\Gamma}{\ell \rho^2}$$

we get

$$\dot{\theta} = \frac{\ell \rho^2}{\Gamma}, \quad \dot{r} = \frac{dr}{d\theta} \dot{\theta} = -\frac{\eta}{\rho^2} \dot{\theta} = -\frac{\ell \eta}{\Gamma},$$

hence

$$\dot{r}^2 + r^2 \dot{\theta}^2 = \frac{\ell^2 (\eta^2 + \rho^2)}{\Gamma^2}.$$

Since

$$\Gamma = \Sigma \left( \ell \sqrt{\eta^2 + \rho^2} \right) = \frac{\ell \sqrt{\eta^2 + \rho^2}}{\varphi^{-1}(\ell \sqrt{\eta^2 + \rho^2})},$$

it follows that

$$\dot{r}^2 + r^2\dot{\theta}^2 = \left[ \varphi^{-1} \left( \ell \sqrt{\eta^2 + \rho^2} \right) \right]^2,$$

and therefore

$$\gamma(\dot{r}^2 + r^2\dot{\theta}^2) = \Sigma \left( \ell \sqrt{\eta^2 + \rho^2} \right) = \Gamma.$$

Thus,

$$r^2\gamma(\dot{r}^2 + r^2\dot{\theta}^2)\dot{\theta} = \frac{1}{\rho^2}\Gamma\frac{\ell\rho^2}{\Gamma} = \ell,$$

so the second equation in (13) is satisfied. Moreover,

$$\dot{r}\gamma(\dot{r}^2 + r^2\dot{\theta}^2) = -\ell\eta,$$

hence, using (25),

$$\frac{d}{dt} \left( \dot{r}\gamma(\dot{r}^2 + r^2\dot{\theta}^2) \right) = -\ell\eta'\dot{\theta} = \ell \left( \rho + \frac{\Gamma}{\ell^2}W'(\rho) \right) \dot{\theta} = \frac{\ell^2\rho^3}{\Gamma} + \rho^2W'(\rho).$$

Since  $r = 1/\rho$  and  $W(\rho) = V(1/\rho)$ , we have  $\rho^2W'(\rho) = -V'(r)$ , therefore the first equation in (13) is also satisfied. Thus  $(r, \theta)$  solves (13), and by Proposition 2.4 the function  $\mathbf{x}(t) = r(t)e^{i\theta(t)}$  is a solution of (2) satisfying (19).  $\square$

Let us write the planar system

$$\rho' = \eta, \quad \eta' = -\rho - \frac{\Gamma(\rho, \eta; \ell)}{\ell^2}W'(\rho), \quad (25)$$

which is equivalent to equation (20). Notice that circular motions  $\mathbf{x}(t) = r_0e^{i\theta(t)}$  of (2) with angular momentum  $\ell$  correspond to equilibria of (25), with  $(\rho_0, \eta_0) = (1/r_0, 0)$ . On the other hand, non-circular motions can be characterized as follows.

**Proposition 2.9.** *Let  $\mathbf{x}(t) = r(t)e^{i\theta(t)}$  be a non-circular solution of (13) and let  $\rho(\theta)$  be as in (19). Then  $\mathbf{x}(t)$  is periodic if and only if  $\rho(\theta)$  is  $\Theta$ -periodic with  $\Theta$  commensurable with  $\pi$ .*

*Proof.* Suppose  $\rho(\theta)$  is a periodic solution of (20) with minimal period  $\Theta$ . Then  $F$  in (24) is also  $\Theta$ -periodic, so

$$t(\theta) = \psi(\theta) + \sigma\theta, \quad \sigma := \frac{1}{\Theta} \int_0^\Theta F(\varphi)d\varphi,$$

where  $\psi$  is  $\Theta$ -periodic. Hence

$$t(\theta + \Theta) = t(\theta) + \sigma\Theta,$$

and the inverse satisfies

$$\theta(t + \sigma\Theta) = \theta(t) + \Theta. \quad (26)$$

Consequently  $r(t) = 1/\rho(\theta(t))$  is  $\sigma\Theta$ -periodic. Moreover, (26) implies that the function  $\vartheta(t) = \theta(t) - t/\sigma$  is  $\sigma\Theta$ -periodic. If  $m\Theta = 2\pi n$  for some positive integers  $m, n$  (i.e.,  $\Theta$  is commensurable with  $\pi$ ), then

$$\theta(t + m\sigma\Theta) = \theta(t) + 2\pi n, \quad r(t + m\sigma\Theta) = r(t),$$

so  $\mathbf{x}(t) = r(t)e^{i\theta(t)}$  is periodic with period  $T = m\sigma\Theta$ .

The inverse implication is carried out similarly.  $\square$

A few more considerations will be useful in the sequel. Let us consider a solution  $\mathbf{x}$  of (2) and compute its angular momentum  $\ell$  and energy  $E$  as in (9) and (10), respectively. Let  $\rho$  be the solution of (20) associated with  $\mathbf{x}$  through Proposition 2.7. Setting  $\eta = \rho'$ , by (7), (10), and (23) we have the energy

$$E = \Psi\left(\ell\sqrt{\rho^2 + \eta^2}\right) + W(\rho). \quad (27)$$

Equivalently,

$$\Psi^{-1}(E - V(1/\rho)) = \ell\sqrt{\rho^2 + (\rho')^2},$$

and so

$$(\rho')^2 = G(\rho; E, \ell), \quad G(\rho; E, \ell) = \frac{[\Psi^{-1}(E - V(1/\rho))]^2}{\ell^2} - \rho^2.$$

Let us now study the equilibria  $(\rho_0, 0)$  of system (25), for which, recalling (21), we must have that

$$P(\rho_0) := \Sigma(\ell\rho_0)W'(\rho_0) + \rho_0\ell^2 = 0.$$

The Jacobian matrix of the vector field driving system (25) is

$$\begin{pmatrix} 0 & 1 \\ -A(\rho, \eta; \ell) & -B(\rho, \eta; \ell) \end{pmatrix},$$

where

$$\begin{aligned} A(\rho, \eta; \ell) &= 1 + \frac{\rho}{\ell\sqrt{\eta^2 + \rho^2}} \Sigma' \left( \ell\sqrt{\eta^2 + \rho^2} \right) W'(\rho) + \frac{1}{\ell^2} \Sigma \left( \ell\sqrt{\eta^2 + \rho^2} \right) W''(\rho), \\ B(\rho, \eta; \ell) &= \frac{\eta}{\ell\sqrt{\eta^2 + \rho^2}} \Sigma' \left( \ell\sqrt{\eta^2 + \rho^2} \right) W'(\rho). \end{aligned}$$

Evaluating them at an equilibrium  $(\rho_0, 0)$ , we get

$$\begin{aligned} A(\rho_0, 0; \ell) &= 1 + \frac{1}{\ell} \Sigma'(\ell\rho_0) W'(\rho_0) + \frac{1}{\ell^2} \Sigma(\ell\rho_0) W''(\rho_0), \\ B(\rho_0, 0; \ell) &= 0. \end{aligned} \quad (28)$$

If  $A_0(\ell) = A(\rho_0, 0; \ell) > 0$  then the equilibrium is a center with associated limit period equal to  $2\pi/\sqrt{A_0(\ell)}$ .

**Remark 2.10.** *The particular potential emphasized in Remark 2.6 is such that  $P(\rho_0) = 0$  and  $A(\rho_0, 0; \ell) = 0$  for every  $\rho_0 > 0$ . In this case all the points  $(\rho_0, 0)$  are degenerate equilibria for (25), and condition (ii) in Definition 1.1 fails. This shows that the naive slogan “all bounded orbits are closed” may hold only for vacuous reasons, and justifies the phase-space local requirement in the B-property.*

*In two concrete cases one can say more, namely that the only bounded motions are the circular ones.*

(a) Relativistic operator. If  $\varphi(s) = \frac{s}{\sqrt{1-s^2}}$ , then (18) gives

$$V'(r) = \frac{\ell^2}{r^2\sqrt{r^2 + \ell^2}}, \quad V(r) = -\frac{\sqrt{r^2 + \ell^2}}{r},$$

and hence  $W(\rho) = V(1/\rho) = -\sqrt{1 + \ell^2\rho^2}$ . Since  $\Psi(s) = \sqrt{1 + s^2}$ , the energy integral (27) becomes

$$E = \sqrt{1 + \ell^2(\rho^2 + \eta^2)} - \sqrt{1 + \ell^2\rho^2} \geq 0,$$

with  $E = 0$  if and only if  $\eta \equiv 0$  (equivalently  $\rho \equiv \text{const}$ ), i.e., the motion is circular. For every  $E > 0$  the level set  $\{E = \text{const}\}$  in the  $(\rho, \eta)$ -plane is the union of the graphs of the two functions

$$\eta = \pm \frac{1}{\ell} \sqrt{E^2 + 2E\sqrt{1 + \ell^2\rho^2}},$$

hence it cannot contain a periodic orbit. Therefore, the only bounded solutions are the circular ones.

(b)  $p$ -Laplacian operator. If  $\varphi(s) = s^{p-1}$ ,  $p > 1$ , and  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ , then (18) yields

$$V'(r) = \frac{\ell^q}{r^{q+1}}, \quad V(r) = -\frac{\ell^q}{q r^q},$$

so that  $W(\rho) = V(1/\rho) = -\frac{\ell^q}{q}\rho^q$  and  $\Psi(s) = \frac{1}{q}s^q$ . Thus (27) reads

$$E = \frac{\ell^q}{q}((\rho^2 + \eta^2)^{q/2} - \rho^q) \geq 0,$$

with  $E = 0$  if and only if  $\eta \equiv 0$ , i.e., the motion is circular. Moreover, for every  $E > 0$  the corresponding energy level is the union of the graphs of the two functions

$$\eta = \pm \sqrt{\left(\rho^q + \frac{qE}{\ell^q}\right)^{2/q} - \rho^2},$$

hence it does not support periodic solutions. Therefore, also in this case the only bounded solutions are the circular ones.

## 2.4 A first consequence of the B-property

Let the B-property hold. For every  $r_0 > 0$ , consider the corresponding circular solution  $\mathbf{x}_0$  of (2), with angular momentum  $\ell_0 = L(r_0)$  as in (14) and energy  $E_0 = E(r_0)$  as in (15). A few considerations are in order.

If, for a given starting time  $t_0$ , we consider a solution  $\mathbf{x}(t)$  of (2) such that  $\mathbf{x}(t_0) = \mathbf{x}_0(t_0)$  and  $\dot{\mathbf{x}}(t_0)$  is slightly different from  $\dot{\mathbf{x}}_0(t_0)$ , but having the same speed, i.e.,  $|\dot{\mathbf{x}}(t_0)| = |\dot{\mathbf{x}}_0(t_0)|$ , then the energy will be the same  $E_0$ , but the angular momentum  $\ell$  will have decreased a bit, hence we will have  $\ell < \ell_0$ . By the B-property, this solution  $\mathbf{x}(t)$  will still be periodic. This is the situation if we keep the same initial speed but slightly change the direction of the velocity.

On the other hand, if as above  $\mathbf{x}(t)$  is still such that  $\mathbf{x}(t_0) = \mathbf{x}_0(t_0)$  and  $\dot{\mathbf{x}}(t_0)$  has a slightly different direction from  $\dot{\mathbf{x}}_0(t_0)$ , but this time we accurately increase the speed, then the angular momentum can be kept the same  $\ell_0$ , but the energy  $E$  will be slightly increased, hence we will have  $E > E_0$ . By the B-property, this solution  $\mathbf{x}(t)$  will still be periodic.

Let us now first keep  $\ell_0$  fixed and vary  $E$  near  $E_0$ . The solution  $\mathbf{x}$  of (2) with angular momentum  $\ell_0$  and energy  $E > E_0$  near  $E_0$  is periodic but not circular. By Proposition 2.9, the corresponding solution  $(\rho(\theta), \eta(\theta))$  of (25) is periodic with period in  $\pi\mathbb{Q}$ . Let us denote by  $\Theta(E, \ell_0)$  its minimal period, defined for  $E > E_0$  in a neighborhood of  $E_0$ . We will call it *apsidal angle*, since it corresponds to the polar angle between two consecutive pericenters in the original system (2). Since  $\Theta(\cdot, \ell_0)$  is a continuous function defined on an interval and its values are in the set  $\pi\mathbb{Q}$ , which is totally disconnected, *this function must be constant*. We can compute this constant by linearizing the equation in  $E_0$ , thus finding that

$$\Theta(E, \ell_0) = \frac{2\pi}{\sqrt{A(1/r_0, 0, L(r_0))}} \in \pi\mathbb{Q},$$

for every  $E > E_0$  in a neighborhood of  $E_0$ .

Let us now vary  $r_0$ . Since the functions  $A$  in (28) and  $L$  in (14) are continuous, and  $\sqrt{A(1/r, 0, L(r))}$  is in  $\mathbb{Q}$  for every  $r$  near  $r_0$ , we deduce that the function  $r \mapsto A(1/r, 0; L(r))$  is constant in a neighborhood of  $r_0$ . Then, since this is true for every  $r_0 > 0$ , this function is constant on the whole interval  $]0, +\infty[$ . We have thus proved the following.

**Proposition 2.11.** *If the B-property holds, there exists  $A_0 > 0$  such that*

$$A(1/r, 0; L(r)) = A_0, \quad \text{for every } r > 0.$$

This property will be useful in what follows next.

**Proposition 2.12.** *If the B-property holds, then the function  $Z(\rho) = \frac{1}{\rho}V'(\frac{1}{\rho})$  is a solution of the differential equation*

$$Z' + \frac{Z}{\rho} \left[ \frac{\Sigma'(\Upsilon(Z))\Upsilon(Z)}{\Sigma(\Upsilon(Z))} + A_0 - 2 \right] = 0. \quad (29)$$

*Proof.* Let us define, recalling (14),

$$\mathcal{L}(\rho) = L(1/\rho) = \frac{1}{\rho} \Upsilon(-W'(\rho)\rho) .$$

As a consequence of (28), we have

$$A_0 - 1 = \frac{1}{\mathcal{L}(\rho)} \Sigma'(\mathcal{L}(\rho)\rho) W'(\rho) + \frac{1}{\mathcal{L}(\rho)^2} \Sigma(\mathcal{L}(\rho)\rho) W''(\rho) ,$$

for every  $\rho > 0$ , and so

$$A_0 - 1 = \frac{\Sigma'(\Upsilon(-W'(\rho)\rho))}{\Upsilon(-W'(\rho)\rho)} [W'(\rho)\rho] + \frac{\Sigma(\Upsilon(-W'(\rho)\rho))}{\Upsilon(-W'(\rho)\rho)^2} [W''(\rho)\rho^2] .$$

Introducing the function  $Z(\rho) = -\rho W'(\rho)$  we get the differential equation

$$A_0 - 1 = -\frac{\Sigma'(\Upsilon(Z))}{\Upsilon(Z)} Z + \frac{\Sigma(\Upsilon(Z))}{\Upsilon(Z)^2} [Z - \rho Z'] ,$$

i.e.,

$$Z' + \frac{Z}{\rho} \left[ \frac{\Sigma'(\Upsilon(Z))\Upsilon(Z)}{\Sigma(\Upsilon(Z))} - 1 \right] + \frac{A_0 - 1}{\rho} \frac{\Upsilon(Z)^2}{\Sigma(\Upsilon(Z))} = 0 .$$

It is now easy to verify that

$$\frac{\Upsilon(Z)^2}{\Sigma(\Upsilon(Z))} = Z ,$$

thus concluding the proof.  $\square$

It will be useful to extend to  $(E_0, \ell_0)$  the function  $\Theta$ , by continuity, setting

$$\Theta(E_0, \ell_0) := \frac{2\pi}{\sqrt{A(1/r_0, 0, \ell_0)}} . \quad (30)$$

### 3 Specialization to the $p$ -Laplacian

In the case of the  $p$ -Laplacian equation (1), i.e., when  $\varphi(s) = s^{p-1}$ , system (25) can be written as

$$\rho' = \eta, \quad \eta' = -\rho - \ell^{-q} (\rho^2 + \eta^2)^{\frac{2-q}{2}} W'(\rho) , \quad (31)$$

where  $W(\rho) = V(1/\rho)$  and  $\ell > 0$  is the (constant) angular momentum. Here, we recall,  $q > 1$  is such that  $(1/p) + (1/q) = 1$ . The energy of a solution is

$$E = \frac{1}{q} \left( \ell \sqrt{\rho^2 + \eta^2} \right)^q + W(\rho) . \quad (32)$$

The differential equation (29) reduces to

$$Z' = \frac{Z}{\rho}(q - A_0).$$

We then easily find its solution  $Z(\rho) = k\rho^{q-A_0}$ , for some constant  $k > 0$ . By Proposition 2.12, recalling that  $V'(r) = r^{-1}Z(1/r)$ , we deduce that a necessary condition for the potential  $V$  to satisfy the B-property is to be such that

$$V'(r) = Cr^{A_0-q-1}, \quad \text{for some } C > 0, \quad (33)$$

where, we recall,  $A_0 > 0$ . Assuming (33), system (31) becomes

$$\rho' = \eta, \quad \eta' = -\rho + C\ell^{-q}(\rho^2 + \eta^2)^{\frac{2-q}{2}}\rho^{q-A_0-1}. \quad (34)$$

We are thus led to consider three cases, according to whether  $A_0 = q$ ,  $A_0 > q$  or  $A_0 < q$ :

- (i) if  $A_0 = q$ , we have  $V(r) = k \ln r$ , for some  $k > 0$ ;
- (ii) if  $A_0 > q$ , we have  $V(r) = kr^\alpha$ , with  $\alpha = A_0 - q$  and  $k > 0$ ;
- (iii) if  $A_0 < q$ , we have  $V(r) = -kr^{-\beta}$ , with  $\beta = q - A_0$  and  $k > 0$ .

We underline the following relation between the constant  $C$  in (34) and the constant  $k$  in the above family

$$C = \begin{cases} k, & \text{in case (i),} \\ \alpha k, & \text{in case (ii),} \\ \beta k, & \text{in case (iii).} \end{cases} \quad (35)$$

Fix  $r_0 > 0$  (e.g.,  $r_0 = 1$ ) and let  $\mathbf{x}_0(t)$  be a circular solution of (1) with radius  $r_0$ , energy  $E_0 = E(r_0)$  and angular momentum  $\ell_0 = L(r_0) > 0$ . Assuming the B-property, the equilibrium point  $(\rho_0, 0) = (1/r_0, 0)$  of system (34) with  $\ell = \ell_0$  is a center surrounded by periodic solutions. We have defined the apsidal angle  $\Theta(E, \ell_0)$  for  $E > E_0$  in a neighborhood of  $E_0$ : it corresponds to the periods of the periodic solutions of (34) in a neighborhood of the equilibrium  $(\rho_0, 0)$ , and we have showed that  $\Theta(E, \ell_0) = 2\pi/\sqrt{A_0} \in \pi\mathbb{Q}$  for every  $E > E_0$  in a neighborhood of  $E_0$ .

Now we want to keep the energy  $E_0$  fixed and vary the angular momentum  $\ell$  below  $\ell_0$ . This can be done following the argument at the beginning of Subsection 2.4, leading to a non-circular periodic solution  $\mathbf{x}_\ell(t)$  of (1) with energy  $E_0$  and angular momentum  $\ell$ . Correspondingly we have a periodic solution  $(\rho, \eta)$  of (34) rotating around the new equilibrium  $(\rho(\ell), 0)$ . By continuity, the period  $\Theta(E_0, \ell)$  of this solution must stay near  $2\pi/\sqrt{A_0}$ , and continuity combined with the fact that this number is in  $\pi\mathbb{Q}$  forces  $\Theta(E_0, \ell) = 2\pi/\sqrt{A_0}$ , as long as it is well defined.

For a solution  $\rho$  of (20), with  $0 < \ell < \ell_0$ , we can find its extremals  $\rho_-(E, \ell)$  and  $\rho_+(E, \ell)$ , and we can write the well known formula

$$\Theta(E, \ell) = 2 \int_{\rho_-(E, \ell)}^{\rho_+(E, \ell)} \frac{d\rho}{\sqrt{G(\rho; E, \ell)}},$$

where, in the three considered cases, we have

$$\begin{aligned} (i) \quad G(\rho; E, \ell) &= \frac{(q(E + k \ln \rho))^{2/q}}{\ell^2} - \rho^2, \\ (ii) \quad G(\rho; E, \ell) &= \frac{(q(E - k\rho^{-\alpha}))^{2/q}}{\ell^2} - \rho^2, \\ (iii) \quad G(\rho; E, \ell) &= \frac{(q(E + k\rho^\beta))^{2/q}}{\ell^2} - \rho^2. \end{aligned}$$

### 3.1 The apsidal angle for very eccentric orbits

Set  $a := q/A_0$ ; by the change of variables

$$u := \ell^a \rho, \quad v := \ell^a \eta, \tag{36}$$

we see that system (34) becomes

$$u' = v, \quad v' = -u + C(u^2 + v^2)^{\frac{2-q}{2}} u^{q-A_0-1}. \tag{37}$$

We distinguish three cases.

#### 3.1.1 The case $A_0 = q$

If  $A_0 = q$  (the case (i)), then  $V(r) = k \ln r$ , so  $W(\rho) = -k \ln \rho$ . Here  $a = 1$ , hence  $u = \ell \rho$ ,  $v = \ell \eta$  and, recalling (32), the energy of a solution is

$$E = \frac{1}{q}(u^2 + v^2)^{q/2} - k \ln u + k \ln \ell.$$

In this case, we define the “modified energy”

$$\mathcal{E} := E - k \ln \ell = \frac{1}{q}(u^2 + v^2)^{q/2} - k \ln u,$$

which is also an integral of motion for (37). Notice that

$$\mathcal{E} \rightarrow +\infty, \quad \text{as } \ell \rightarrow 0^+.$$

Taking from the beginning  $r_0 = 1$ , we find  $E_0 = k/q$  and  $\ell_0 = k^{1/q}$ . The only equilibrium is  $(u_0, 0)$ , with  $u_0 = k^{1/q}$ . Solving the equation

$$\frac{1}{q}u^q - k \ln u = \mathcal{E},$$

we find the extreme points  $u_-(\mathcal{E}) < u_0 < u_+(\mathcal{E})$  of the solution  $u$ . We notice that

$$\lim_{\mathcal{E} \rightarrow +\infty} u_-(\mathcal{E}) = 0, \quad \lim_{\mathcal{E} \rightarrow +\infty} u_+(\mathcal{E}) = +\infty.$$

The apsidal angle can then be written as

$$\Theta(\mathcal{E}) = 2 \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{(q(\mathcal{E} + k \ln u))^{2/q} - u^2}}.$$

It is well defined for every  $\mathcal{E} > \frac{k}{q}(1 - \ln k)$ .

Our aim here is to prove the following.

**Proposition 3.1.** *We have that*

$$\lim_{\ell \rightarrow 0^+} \Theta(E_0, \ell) = \pi. \quad (38)$$

*Proof.* We can write

$$\Theta(E_0, \ell) = \Theta(\mathcal{E}) := 2 \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}}, \quad H_{\mathcal{E}}(u) := (q(\mathcal{E} + k \ln u))^{2/q} - u^2.$$

Hence we need to show that

$$\lim_{\mathcal{E} \rightarrow +\infty} \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \frac{\pi}{2}.$$

We write

$$\int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_{u_-(\mathcal{E})}^1 \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} + \int_1^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}}.$$

By a change of variable, we have that

$$\int_{u_-(\mathcal{E})}^1 \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_1^{u_-(\mathcal{E})^{-1}} \frac{ds}{\sqrt{(1 + qk u_-(\mathcal{E})^{-q} \ln s)^{2/q} - s^2}}.$$

We need the following.

**Lemma 3.2.** *For any constants  $q > 1$  and  $B > 0$ ,*

$$\lim_{y \rightarrow +\infty} \int_1^y \frac{ds}{\sqrt{(1 + By^q \ln s)^{2/q} - s^2}} = 0.$$

By the use of this result, we have that

$$\lim_{\mathcal{E} \rightarrow +\infty} \int_{u_-(\mathcal{E})}^1 \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = 0. \quad (39)$$

On the other hand,

$$\int_1^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_{u_+(\mathcal{E})^{-1}}^1 \frac{ds}{\sqrt{(1 + qk u_+(\mathcal{E})^{-q} \ln s)^{2/q} - s^2}}.$$

We need the following.

**Lemma 3.3.** *For any constants  $q > 1$  and  $B > 0$ ,*

$$\lim_{y \rightarrow 0^+} \int_y^1 \frac{ds}{\sqrt{(1 + By^q \ln s)^{2/q} - s^2}} = \frac{\pi}{2}.$$

As a consequence,

$$\lim_{\mathcal{E} \rightarrow +\infty} \int_1^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \frac{\pi}{2}. \quad (40)$$

The proofs of Lemmas 3.2 and 3.3 are postponed to Appendix B. Adding the two limits in (39) and (40), the proposition is thus settled.  $\square$

We thus have the following consequence.

**Proposition 3.4.** *If the B-property holds and  $A_0 = q$ , then necessarily  $q = 4$ .*

*Proof.* Having fixed  $E_0$ , when we vary  $\ell \in ]0, \ell_0[$  the apsidal angle  $\Theta(E_0, \ell)$  remains constant. Then, by (30) and (38), it has to be that  $2\pi/\sqrt{q} = 2\pi/\sqrt{A_0} = \pi$ , hence  $q = 4$ .  $\square$

### 3.1.2 The case $A_0 > q$

If  $A_0 > q$ , we have  $V(r) = kr^\alpha$ , with  $k > 0$  and  $\alpha = A_0 - q \in ]0, A_0[$ . Taking  $r_0 = 1$ , we find  $E_0 = kA_0/q$  and  $\ell_0 = (\alpha k)^{1/q}$ . Recalling (31) and (36), the energy of the circular solution is

$$E_0 = \ell^{\frac{q(A_0-q)}{A_0}} \left[ \frac{1}{q} (u^2 + v^2)^{q/2} + k u^{q-A_0} \right].$$

In this case, we define the “modified energy”

$$\mathcal{E} := E_0 \ell^{-\frac{q(A_0-q)}{A_0}} = \frac{1}{q} (u^2 + v^2)^{q/2} + k u^{q-A_0},$$

which is an integral of motion for (37). Then,

$$\mathcal{E} \rightarrow +\infty, \text{ as } \ell \rightarrow 0^+.$$

The only equilibrium is  $(u_0, 0)$ , with  $u_0 = (k\alpha)^{1/A_0}$ . Solving the equation

$$\frac{1}{q}u^q + ku^{-\alpha} = \mathcal{E},$$

we find the extreme points  $u_-(\mathcal{E}) < u_0 < u_+(\mathcal{E})$  of the solution  $u$ . We notice that

$$\lim_{\mathcal{E} \rightarrow +\infty} u_-(\mathcal{E}) = 0, \quad \lim_{\mathcal{E} \rightarrow +\infty} u_+(\mathcal{E}) = +\infty.$$

The apsidal angle can then be written as

$$\Theta(\mathcal{E}) = 2 \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{(q(\mathcal{E} - ku^{-\alpha}))^{2/q} - u^2}}.$$

It is well defined for every  $\mathcal{E} > \mathcal{E}_0 := (k\alpha)^{q/A_0}(q^{-1} + \alpha^{-1})$ .

Our aim here is to prove the following.

**Proposition 3.5.** *We have that*

$$\lim_{\ell \rightarrow 0^+} \Theta(E_0, \ell) = \pi. \quad (41)$$

*Proof.* We want to rewrite the formula of the apsidal angle using the variables  $(u, v)$  instead of  $(\rho, \eta)$ . Recall that  $\mathcal{E} = E_0 \ell^{-\frac{q\alpha}{q+\alpha}}$  and notice that

$$u_{\pm}(\mathcal{E}) = \ell^{q/(q+\alpha)} \rho_{\pm}(E_0, \ell).$$

Then, we can write

$$\Theta(E_0, \ell) = \Theta(\mathcal{E}) := 2 \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}}, \quad H_{\mathcal{E}}(u) := (q(\mathcal{E} - ku^{-\alpha}))^{2/q} - u^2.$$

Hence we need to show that

$$\lim_{\mathcal{E} \rightarrow +\infty} \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \frac{\pi}{2}.$$

As in the proof of Proposition 3.1,

$$\int_{u_-(\mathcal{E})}^1 \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_1^{u_-(\mathcal{E})^{-1}} \frac{ds}{\sqrt{(1 + qk u_-(\mathcal{E})^{-A_0}(1 - s^{-\alpha}))^{2/q} - s^2}},$$

and we need the following.

**Lemma 3.6.** *For any constants  $q > 1$ ,  $B > 0$  and  $A > q$ ,*

$$\lim_{y \rightarrow +\infty} \int_1^y \frac{ds}{\sqrt{(1 + By^A(1 - s^{q-A}))^{2/q} - s^2}} = 0.$$

By the use of this result, we have that

$$\lim_{\mathcal{E} \rightarrow +\infty} \int_{u_-(\mathcal{E})}^1 \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = 0. \quad (42)$$

On the other hand,

$$\int_1^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_{u_+(\mathcal{E})^{-1}}^1 \frac{ds}{\sqrt{(1 + qk u_+(\mathcal{E})^{-A_0}(1 - s^{-\alpha}))^{2/q} - s^2}}.$$

We need the following.

**Lemma 3.7.** *For any constants  $q > 1$ ,  $B > 0$  and  $A > q$ ,*

$$\lim_{y \rightarrow 0^+} \int_y^1 \frac{ds}{\sqrt{(1 + By^A(1 - s^{q-A}))^{2/q} - s^2}} = \frac{\pi}{2}.$$

We then deduce that

$$\lim_{\mathcal{E} \rightarrow +\infty} \int_1^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \frac{\pi}{2}. \quad (43)$$

The proofs of Lemmas 3.6 and 3.7 are postponed to Appendix B. Adding the two limits in (42) and (43), the proposition is thus proved.  $\square$

We thus have the following consequence.

**Proposition 3.8.** *If the B-property holds and  $A_0 = q + \alpha$ , then necessarily  $A_0 = 4$ , i.e.,  $\alpha = 4 - q$ .*

*Proof.* Having fixed  $E_0$ , when we vary  $\ell \in ]0, \ell_0[$  the apsidal angle  $\Theta(E_0, \ell)$  remains constant. Then, by (30) and (41), it has to be that  $2\pi/\sqrt{q + \alpha} = 2\pi/\sqrt{A_0} = \pi$ , hence  $q + \alpha = 4$ .  $\square$

**Remark 3.9.** *Notice that, if  $p = q = 2$ , the above proposition gives  $\alpha = 2$ , which corresponds to the harmonic oscillator.*

### 3.1.3 The case $A_0 < q$

If  $A_0 < q$ , we have  $V(r) = -kr^{-\beta}$ , with  $k > 0$  and  $\beta = q - A_0 \in ]0, q[$ . Taking  $r_0 = 1$ , we find  $E_0 = -kA_0/q$  and  $\ell_0 = (\beta k)^{1/q}$ . We proceed analogously as for the case  $A_0 > q$  and we define the “modified energy”

$$\mathcal{E} := E_0 \ell^{-\frac{q(A_0 - q)}{A_0}} = \frac{1}{q}(u^2 + v^2)^{q/2} - k u^{q - A_0}.$$

However, this time, we see that

$$\mathcal{E} \rightarrow 0^-, \text{ as } \ell \rightarrow 0^+.$$

The only equilibrium is  $(u_0, 0)$ , with  $u_0 = (k\beta)^{1/A_0}$ . Solving the equation

$$\frac{1}{q}u^q - ku^\beta = \mathcal{E},$$

we find the extreme points  $u_-(\mathcal{E}) < u_0 < u_+(\mathcal{E})$  of the solution  $u$ . We notice that

$$\lim_{\mathcal{E} \rightarrow 0^-} u_-(\mathcal{E}) = 0, \quad \lim_{\mathcal{E} \rightarrow 0^-} u_+(\mathcal{E}) = U_0 := (kq)^{1/A_0}.$$

The apsidal angle can then be written as

$$\Theta(\mathcal{E}) = 2 \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{(q(\mathcal{E} + ku^\beta))^{2/q} - u^2}}.$$

It is well defined for every  $\mathcal{E} \in ]\mathcal{E}_0, 0[$ , with  $\mathcal{E}_0 := (k\beta)^{q/A_0}(q^{-1} - \beta^{-1}) < 0$ .

Our aim here is to prove the following.

**Proposition 3.10.** *We have that*

$$\lim_{\ell \rightarrow 0^+} \Theta(E_0, \ell) = \frac{\pi q}{q - \beta} = \frac{\pi q}{A_0}. \quad (44)$$

*Proof.* As seen above, we can write

$$\Theta(E_0, \ell) = \Theta(\mathcal{E}) := 2 \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}}, \quad H_{\mathcal{E}}(u) := (q(\mathcal{E} + ku^\beta))^{2/q} - u^2.$$

Hence we need to show that

$$\lim_{\mathcal{E} \rightarrow 0^-} \int_{u_-(\mathcal{E})}^{u_+(\mathcal{E})} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \frac{\pi q}{2A_0}.$$

Set  $u_0 = (k\beta)^{1/A_0}$  (the equilibrium), and write

$$\int_{u_-(\varepsilon)}^{u_+(\varepsilon)} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_{u_-(\varepsilon)}^{u_0} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} + \int_{u_0}^{u_+(\varepsilon)} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}}.$$

Changing the variable, we have that

$$\int_{u_-(\varepsilon)}^{u_0} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_1^{u_-(\varepsilon)^{-1}u_0} \frac{ds}{\sqrt{(1 - qk u_-(\varepsilon)^{-A_0}(1 - s^{-\beta}))^{2/q} - s^2}}.$$

We need the following.

**Lemma 3.11.** *For any constants  $q > 1$  and  $A \in ]0, q[$ ,*

$$\lim_{y \rightarrow +\infty} \int_1^y \frac{ds}{\sqrt{(1 - qy^A(1 - s^{q-A}))^{2/q} - s^2}} = \frac{q}{A} \arcsin(q^{-1/q}).$$

By the use of this result, we have that

$$\lim_{\varepsilon \rightarrow +\infty} \int_{u_-(\varepsilon)}^1 \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \frac{q}{A_0} \arcsin(q^{-1/q}). \quad (45)$$

On the other hand,

$$\int_{u_0}^{u_+(\varepsilon)} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \int_{u_+(\varepsilon)^{-1}u_0}^1 \frac{ds}{\sqrt{(1 - qk u_+(\varepsilon)^{-A_0}(1 - s^{-\beta}))^{2/q} - s^2}}.$$

We need the following.

**Lemma 3.12.** *For any constants  $q > 1$  and  $A \in ]0, q[$ , setting  $y_0 := q^{-1/A}$ , we have that*

$$\lim_{y \rightarrow y_0} \int_y^1 \frac{ds}{\sqrt{(1 - qy^A(1 - s^{q-A}))^{2/q} - s^2}} = \frac{q}{A} \left( \frac{\pi}{2} - \arcsin(q^{-1/q}) \right).$$

Because of this,

$$\lim_{\varepsilon \rightarrow +\infty} \int_{u_0}^{u_+(\varepsilon)} \frac{du}{\sqrt{H_{\mathcal{E}}(u)}} = \frac{q}{A_0} \left( \frac{\pi}{2} - \arcsin(q^{-1/q}) \right). \quad (46)$$

The proofs of Lemmas 3.11 and 3.12 are postponed to Appendix B. Adding the two limits in (45) and (46), the proof is completed.  $\square$

We thus have the following consequence.

**Proposition 3.13.** *If the B-property holds and  $A_0 = q - \beta$ , then necessarily  $\beta = q - q^2/4$ .*

*Proof.* Having fixed  $E_0$ , when we vary  $\ell \in ]0, \ell_0[$  the apsidal angle  $\Theta(E_0, \ell)$  remains constant. Hence, by (30) and (44), it has to be that  $2\pi/\sqrt{A_0} = \pi q/A_0$ , hence  $\beta = q - q^2/4$ .  $\square$

**Remark 3.14.** *Notice that, if  $p = q = 2$ , the above proposition gives  $\beta = 1$ , which corresponds to the Keplerian potential.*

### 3.2 Small-amplitude expansion near circular orbits

In the three potential families arising from (33), system (37) admits a first integral of the form

$$\mathcal{E} = \frac{1}{q}(u^2 + v^2)^{q/2} + U(u), \quad (47)$$

where, in the three possible cases,

$$\begin{aligned} \text{(i) } A_0 = q : & \quad U(u) = -k \ln u; \\ \text{(ii) } A_0 > q : & \quad U(u) = k u^{q-A_0} = k u^{-\alpha}; \\ \text{(iii) } A_0 < q : & \quad U(u) = -k u^{q-A_0} = -k u^\beta. \end{aligned} \quad (48)$$

Fixing an energy level  $\mathcal{E}$  and taking  $v = u'$  in (47), we obtain

$$(u')^2 = \tilde{G}(u; \mathcal{E}), \quad \tilde{G}(u; \mathcal{E}) := \left( q(\mathcal{E} - U(u)) \right)^{2/q} - u^2. \quad (49)$$

The turning points of the oscillation are the positive roots  $u_-(\mathcal{E}) < u_+(\mathcal{E})$  of  $\tilde{G}(\cdot; \mathcal{E})$ .

A circular motion of (1) corresponds to an equilibrium  $(u_0, 0)$  of (37), where  $u_0 = C^{1/A_0}$ , with  $C$  as in (35). It lies on the energy level

$$\mathcal{E}_0 = \frac{1}{q}u_0^q + U(u_0),$$

and it is a constant solution of (49) satisfying

$$\tilde{G}(u_0; \mathcal{E}_0) = 0 \quad \text{and} \quad \partial_u \tilde{G}(u_0; \mathcal{E}_0) = 0.$$

Differentiating (49) at points where  $u' \neq 0$  yields  $2u'u'' = (\partial_u \tilde{G}(u; \mathcal{E}))u'$ , hence

$$u'' = \frac{1}{2} \partial_u \tilde{G}(u; \mathcal{E}),$$

and the identity extends by continuity at the turning points.

We will now use the Lindstedt–Poincaré technique in order to analyze the periodic solutions  $u$  near the equilibrium  $u_0$  (see, e.g., [8, §7.3] for the details). Writing  $u = u_0 + w$  and expanding  $\partial_u \tilde{G}(u_0 + w)$  at  $w = 0$  we obtain the nonlinear equation

$$w'' + \omega^2 w + c_2 w^2 + c_3 w^3 = O(w^4), \quad (50)$$

with

$$\omega^2 = -\frac{1}{2} \partial_u^2 \tilde{G}(u_0; \mathcal{E}), \quad c_2 = -\frac{1}{4} \partial_u^3 \tilde{G}(u_0; \mathcal{E}), \quad c_3 = -\frac{1}{12} \partial_u^4 \tilde{G}(u_0; \mathcal{E}). \quad (51)$$

We already know that  $\omega^2 = A_0$ , so we need to compute  $c_2$  and  $c_3$ . Having fixed  $\mathcal{E}$ , let us use the notation  $\aleph(u) = (q(\mathcal{E} - U(u)))^{\frac{1}{q}}$ . Then,

$$\begin{aligned}
\partial_u \tilde{G}(u; \mathcal{E}) &= -2U'(u) \aleph(u)^{2-q} - 2u, \\
\partial_u^2 \tilde{G}(u; \mathcal{E}) &= -2U''(u) \aleph(u)^{2-q} + 2(2-q)U'(u)^2 \aleph(u)^{2-2q} - 2, \\
\partial_u^3 \tilde{G}(u; \mathcal{E}) &= -2U'''(u) \aleph(u)^{2-q} \\
&\quad + 6(2-q)U'(u)U''(u) \aleph(u)^{2-2q} \\
&\quad - 2(2-q)(2-2q)U'(u)^3 \aleph(u)^{2-3q}, \\
\partial_u^4 \tilde{G}(u; \mathcal{E}) &= -2U''''(u) \aleph(u)^{2-q} \\
&\quad + 8(2-q)U'(u)U'''(u) \aleph(u)^{2-2q} \\
&\quad + 6(2-q)U''(u)^2 \aleph(u)^{2-2q} \\
&\quad - 12(2-q)(2-2q)U'(u)^2 U''(u) \aleph(u)^{2-3q} \\
&\quad + 2(2-q)(2-2q)(2-3q)U'(u)^4 \aleph(u)^{2-4q}.
\end{aligned}$$

Notice that, at the equilibrium  $u_0 = C^{1/A_0}$ , we have  $\aleph(u_0) = (q(\mathcal{E}_0 - U(u_0)))^{\frac{1}{q}} = u_0$ .

Since  $U'(u) = -Cu^{q-A_0-1}$  in all the three cases, setting  $\hat{\alpha} = A_0 - q$  we have

$$\begin{aligned}
\partial_u^3 \tilde{G}(u_0; \mathcal{E}) &= u_0^{-1} [2(\hat{\alpha} + 1)(\hat{\alpha} + 2) - 6(2-q)(\hat{\alpha} + 1) + 2(2-q)(2-2q)], \\
\partial_u^4 \tilde{G}(u_0; \mathcal{E}) &= u_0^{-2} [-2(\hat{\alpha} + 1)(\hat{\alpha} + 2)(\hat{\alpha} + 3) + 8(2-q)(\hat{\alpha} + 1)(\hat{\alpha} + 2) + 6(2-q)(\hat{\alpha} + 1)^2 \\
&\quad - 12(2-q)(2-2q)(\hat{\alpha} + 1) + 2(2-q)(2-2q)(2-3q)].
\end{aligned}$$

Recalling (51), a careful computation gives

$$\begin{aligned}
c_2 &= -\frac{1}{4} \partial_u^3 \tilde{G}(u_0; \mathcal{E}) = \frac{A_0(3-q-A_0)}{2u_0}, \\
c_3 &= -\frac{1}{12} \partial_u^4 \tilde{G}(u_0; \mathcal{E}) = \frac{A_0(A_0^2 + 4A_0q - 8A_0 + q^2 - 2q - 1)}{6u_0^2}.
\end{aligned}$$

Setting  $\varepsilon = (\max |w|)/u_0$ , we now write the expansion of the apsidal angle  $\Theta$ , which is the minimal period of  $u$ . In the notation of [8, p. 165] (see Appendix A), the *nonlinear frequency*  $\Omega = 2\pi/\Theta$  can be written as

$$\Omega = \omega(1 - \mathcal{K}\varepsilon^2 + O(\varepsilon^4)),$$

with

$$\mathcal{K} = \left( \frac{5c_2^2}{12\omega^4} - \frac{3c_3}{8\omega^2} \right) u_0^2. \quad (52)$$

Therefore our expansion is

$$\Theta = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega} \left( 1 + \mathcal{K} \varepsilon^2 + O(\varepsilon^4) \right).$$

We now compute  $\mathcal{K}$  for the three families of potentials  $U$  in (48), using the values in (51):

$$\begin{aligned} \mathcal{K} = \mathcal{K}(q, A_0) &= \frac{5}{12} \left( \frac{3 - q - A_0}{2} \right)^2 - \frac{3}{8} \frac{A_0^2 + 4A_0q - 8A_0 + q^2 - 2q - 1}{6} \\ &= \frac{1}{24} [A_0^2 - A_0q - 3A_0 + q^2 - 12q + 24]. \end{aligned}$$

The B-property forces  $\Theta$  to be locally constant in a neighborhood of each circular orbit, hence necessarily  $\mathcal{K} = 0$ . Let us analyze the three different cases in (48).

(i) *Logarithmic case (i.e.,  $U(u) = -k \ln u$ ).* In this case, since  $A_0 = q$ , we have

$$\mathcal{K}(q, q) = \frac{q^2 - 15q + 24}{24}.$$

By Proposition 3.4 we simply need to consider the case  $q = 4$ , which gives  $\mathcal{K}(4, 4) = -5/6 \neq 0$ , a contradiction. This case is thus excluded.

(ii) *Positive power case  $A_0 > q$  (i.e.,  $U(u) = k u^{-\alpha}$ , with  $\alpha = A_0 - q > 0$ ).* By Proposition 3.8, the substitution  $A_0 = \alpha + q = 4$  gives

$$\mathcal{K}(q, 4) = \frac{(q - 2)(q - 14)}{24}.$$

Since  $\alpha = 4 - q > 0$  forces  $q < 4$ , we have that  $\mathcal{K}(q, 4) = 0$  if and only if  $q = 2$  (i.e.,  $p = 2$ ), yielding  $\alpha = 2$ , namely the isotropic harmonic oscillator.

(iii) *Negative power case  $A_0 < q$  (i.e.,  $U(u) = -k u^\beta$ , with  $\beta = q - A_0 > 0$ ).* By Proposition 3.13 we necessarily have  $\beta = q - \frac{q^2}{4}$ , hence  $q < 4$  and  $A_0 = q^2/4$ . Substituting, we get

$$\mathcal{K}(q, q^2/4) = \frac{(q - 2)(q^3 - 2q^2 - 192)}{384}.$$

The cubic factor is negative for  $q \in ]1, 4[$ , hence  $\mathcal{K}(q, q^2/4) = 0$  if and only if  $q = 2$  (i.e.,  $p = 2$ ), and correspondingly  $\beta = 1$ , namely the Kepler potential.

The proof of Theorem 1.2 is thus complete.

## A Appendix: the Lindstedt–Poincaré technique

For the reader's convenience we provide here a self-contained derivation of the value of the constant  $\mathcal{K}$  given in in (52). The computation is a standard application of the Lindstedt–Poincaré technique, as explained in [8, §7.3].

Consider the nonlinear equation

$$\ddot{x} + \omega^2 x + c_2 x^2 + c_3 x^3 = O(x^4), \quad (53)$$

where the nonlinearity is given as a Taylor expansion at  $x = 0$ . Let us focus on a periodic solution  $x$  of small amplitude  $|x| \sim a$ . We assume that such a solution has frequency

$$\Omega = \omega + \nu_1 a + \nu_2 a^2 + O(a^3), \quad (54)$$

and introduce the time rescaling  $\tau = \Omega t$ . So, equation (53) in the variable  $\tau$  becomes

$$\Omega^2 x'' + \omega^2 x + c_2 x^2 + c_3 x^3 = O(x^4). \quad (55)$$

We then consider the following Taylor expansion of the solution  $x$  with respect to the  $\tau$  variable, i.e.,

$$x(\tau) = ax_1(\tau) + a^2 x_2(\tau) + a^3 x_3(\tau) + O(a^4). \quad (56)$$

From (54) we get

$$\Omega^2 = (\omega + \nu_1 a + \nu_2 a^2 + O(a^3))^2 = \omega^2 + 2\omega\nu_1 a + (\nu_1^2 + 2\omega\nu_2)a^2 + O(a^3). \quad (57)$$

Substituting (56) and (57) in (55), we obtain

$$\begin{aligned} a(\omega^2 x_1'' + \omega^2 x_1) + a^2(\omega^2 x_2'' + 2\omega\nu_1 x_1 + \omega^2 x_2 + c_2 x_1^2) \\ + a^3(\omega^2 x_3'' + 2\omega\nu_1 x_2'' + (2\nu_2\omega + \nu_1^2)x_1'' + \omega^2 x_3 + 2c_2 x_1 x_2 + c_3 x_1^3) + O(a^4) = 0. \end{aligned}$$

At the first order we get  $x_1'' + x_1 = 0$ , thus giving us

$$x_1(\tau) = \cos \tau$$

(notice that the choice of the phase is free since the equation is autonomous), while at the second order we have

$$\omega^2(x_2'' + x_2) = -2\omega\nu_1 x_1 - c_2 x_1^2.$$

Since by assumption the solution  $x$  is periodic, the *resonant* term  $2\omega\nu_1 x_1 = 2\omega\nu_1 \cos \tau$  must be equal to zero, hence  $\nu_1 = 0$ . So,

$$x_2'' + x_2 = -\frac{c_2}{2\omega^2} (1 + \cos 2\tau),$$

whose solution is

$$x_2(\tau) = -\frac{c_2}{2\omega^2} + \frac{c_2}{6\omega^2} \cos 2\tau.$$

At the third order, we need to solve

$$\omega^2(x_3'' + x_3) = -2\omega\nu_2 x_1'' - 2c_2 x_1 x_2 - c_3 x_1^3,$$

which corresponds to

$$x_3'' + x_3 = \frac{2\nu_2}{\omega} \cos \tau + \frac{c_2^2}{\omega^4} \cos \tau \left(1 - \frac{1}{3} \cos 2\tau\right) - \frac{c_3}{\omega^2} (\cos \tau)^3.$$

Using  $\cos^3 \tau = \frac{1}{4}(3 \cos \tau + \cos 3\tau)$  and  $\cos \tau \cos 2\tau = \frac{1}{2}(\cos 3\tau + \cos \tau)$ , we get

$$x_3'' + x_3 = \left[ \frac{2\nu_2}{\omega} + \frac{5}{6} \frac{c_2^2}{\omega^4} + \frac{3}{4} \frac{c_3}{\omega^2} \right] \cos \tau + \left[ -\frac{1}{6} \frac{c_2^2}{\omega^4} + \frac{1}{4} \frac{c_3}{\omega^2} \right] \cos(3\tau).$$

Again, the *resonant* term with  $\cos \tau$  must be canceled out, thus giving us

$$\nu_2 = \frac{3c_3}{8\omega} - \frac{5c_2^2}{12\omega^3}.$$

Substituting this into (54) provides us the relation

$$\Omega = \omega \left( 1 - \left[ \frac{5c_2^2}{12\omega^4} - \frac{3c_3}{8\omega^2} \right] a^2 + O(a^4) \right). \quad (58)$$

In Section 3.2 we set  $u = u_0 + w$  and obtain (50) with coefficients  $\omega, c_2, c_3$  given by (51). Writing the oscillation amplitude as  $a := \max |w|$  and  $\varepsilon := a/u_0$ , formula (58) yields

$$\Omega = \omega \left( 1 - \left[ \frac{5c_2^2}{12\omega^4} - \frac{3c_3}{8\omega^2} \right] a^2 + O(a^4) \right) = \omega \left( 1 - \mathcal{K} \varepsilon^2 + O(\varepsilon^4) \right),$$

where

$$\mathcal{K} = \left( \frac{5c_2^2}{12\omega^4} - \frac{3c_3}{8\omega^2} \right) u_0^2,$$

which is precisely (52).

## B Appendix: Proof of the lemmas

In this appendix we provide the proofs of the lemmas stated in Section 3.

### B.1 Proof of Lemma 3.2

Given  $q > 1$ ,  $B > 0$ , we choose  $y$  sufficiently large so to have

$$s_0(y) := \exp\left(\frac{1}{By^q}\right) < e, \quad (59)$$

and split the integral

$$I(y) := \int_1^y \frac{ds}{\sqrt{(1 + By^q \ln s)^{2/q} - s^2}}$$

into three parts  $I(y) = I_1(y) + I_2(y) + I_3(y)$ , with

$$\begin{aligned} I_1(y) &:= \int_1^{s_0(y)} \frac{ds}{\sqrt{(1 + By^q \ln s)^{\frac{2}{q}} - s^2}}, \\ I_2(y) &:= \int_{s_0(y)}^e \frac{ds}{\sqrt{(1 + By^q \ln s)^{\frac{2}{q}} - s^2}}, \\ I_3(y) &:= \int_e^y \frac{ds}{\sqrt{(1 + By^q \ln s)^{\frac{2}{q}} - s^2}}. \end{aligned}$$

We will show that  $\lim_{y \rightarrow +\infty} I_k(y) = 0$  for  $k = 1, 2, 3$ .

*Estimate of  $I_1(y)$ .* Using the change of variables

$$t = By^q \ln s, \quad s = e^{t/(By^q)}, \quad (60)$$

we get, using (59),

$$I_1(y) = \frac{1}{By^q} \int_0^1 \frac{e^{t/(By^q)} dt}{\sqrt{(1+t)^{\frac{2}{q}} - e^{2t/(By^q)}}} \leq \frac{e}{By^q} \int_0^1 \frac{dt}{\sqrt{(1+t)^{\frac{2}{q}} - e^{2t/(By^q)}}}.$$

The inequalities

$$\begin{aligned} e^s &\leq 1 + 2s, \quad \text{for every } s \in [0, 1], \\ (1+t)^{\frac{2}{q}} &\geq 1 + c_q t, \quad \text{for every } t \in [0, 1], \quad \text{where } c_q := \min\{1, 2^{\frac{2}{q}} - 1\} > 0, \end{aligned}$$

permit us to provide the following estimate for  $y$  large:

$$(1+t)^{\frac{2}{q}} - e^{2t/(By^q)} \geq (1 + c_q t) - \left(1 + \frac{4t}{By^q}\right) = t \left(c_q - \frac{4}{By^q}\right) \geq \frac{c_q}{2} t.$$

Thus, for a suitable constant  $C > 0$ ,

$$I_1(y) \leq \frac{e}{By^q} \int_0^1 \frac{dt}{\sqrt{(c_q/2)t}} = \frac{C}{y^q} \xrightarrow{y \rightarrow \infty} 0.$$

*Estimate of  $I_2(y)$ .* We use again the change of variable in (60) and get similarly

$$I_2(y) = \frac{1}{By^q} \int_1^{By^q} \frac{e^{t/(By^q)} dt}{\sqrt{(1+t)^{\frac{2}{q}} - e^{2t/(By^q)}}} \leq \frac{e}{By^q} \int_1^{By^q} \frac{dt}{\sqrt{(1+t)^{\frac{2}{q}} - e^{2t/(By^q)}}}.$$

Choose  $T_0 > 1$  such that  $(1+t)^{\frac{2}{q}} \geq 2e^2$  for every  $t \geq T_0$ . We write  $I_2(y) = J_2^{(a)}(y) + J_2^{(b)}(y)$ , with

$$J_2^{(a)}(y) := \frac{e}{By^q} \int_1^{T_0} \frac{dt}{\sqrt{(1+t)^{\frac{2}{q}} - e^{2t/(By^q)}}},$$

$$J_2^{(b)}(y) := \frac{e}{By^q} \int_{T_0}^{By^q} \frac{dt}{\sqrt{(1+t)^{\frac{2}{q}} - e^{2t/(By^q)}}},$$

assuming  $y$  large enough so to have  $T_0 < By^q$ . Let us first estimate  $J_2^{(b)}(y)$ . We have

$$(1+t)^{\frac{2}{q}} - e^{2t/(By^q)} \geq (1+t)^{\frac{2}{q}} - e^2 \geq \frac{1}{2}(1+t)^{\frac{2}{q}}, \quad \text{for every } t \in [T_0, By^q],$$

hence, for some suitable positive constants  $C_1$  and  $C_2$ ,

$$J_2^{(b)}(y) \leq \frac{e\sqrt{2}}{By^q} \int_{T_0}^{By^q} (1+t)^{-\frac{1}{q}} dt \leq \frac{C_1}{y^q} (By^q)^{1-1/q} = C_2 y^{-1} \xrightarrow{y \rightarrow \infty} 0.$$

Concerning the integral  $J_2^{(a)}(y)$ , we first observe that  $(1+t)^{\frac{2}{q}} \geq 2^{\frac{2}{q}}$ , for every  $t \in [1, T_0]$ . Setting  $\eta = \frac{1}{2}(2^{\frac{2}{q}} - 1)$ , since  $e^{2T_0/(By^q)} \rightarrow 1$  as  $y \rightarrow +\infty$ , for  $y$  sufficiently large and  $t \in [1, T_0]$  we get

$$(1+t)^{\frac{2}{q}} - e^{\frac{2t}{By^q}} \geq 1 + 2\eta - e^{\frac{2t}{By^q}} > \eta,$$

and consequently, for a suitable constant  $C > 0$ ,

$$J_2^{(a)}(y) \leq \frac{e}{By^q} \int_1^{T_0} \frac{dt}{\sqrt{\eta}} = \frac{C}{y^q} \xrightarrow{y \rightarrow \infty} 0.$$

*Estimate of  $I_3(y)$ .* Setting  $s = yr$ , we get

$$I_3(y) = \int_{e/y}^1 \frac{dr}{\sqrt{(B \ln(yr) + y^{-q})^{2/q} - r^2}}.$$

Once more we split the integral, for  $y$  sufficiently large, as  $I_3(y) = K_3^{(a)}(y) + K_3^{(b)}(y)$ , where

$$K_3^{(a)}(y) := \int_{e/y}^{y^{-1/2}} \frac{dr}{\sqrt{(B \ln(yr) + y^{-q})^{2/q} - r^2}},$$

$$K_3^{(b)}(y) := \int_{y^{-1/2}}^1 \frac{dr}{\sqrt{(B \ln(yr) + y^{-q})^{2/q} - r^2}}.$$

Concerning  $K_3^{(a)}(y)$ , we note that, since  $\ln(yr) \geq 1$  and  $r^2 \leq y^{-1}$  for  $r \in [e/y, y^{-1/2}]$ ,

$$(B \ln(yr) + y^{-q})^{2/q} - r^2 \geq B^{2/q} - y^{-1} \geq \frac{1}{2} B^{2/q}, \quad \text{for every } r \in [e/y, y^{-1/2}].$$

So the integrand is bounded by a constant  $C > 0$  and we get

$$K_3^{(a)}(y) \leq C (y^{-1/2} - ey^{-1}) \leq C y^{-1/2} \xrightarrow{y \rightarrow \infty} 0.$$

We now focus our attention on  $K_3^{(b)}(y)$  and notice that for  $r \in [y^{-1/2}, 1]$  we have  $\ln(yr) \geq \frac{1}{2} \ln y$ , hence, for  $y$  sufficiently large,

$$(B \ln(yr) + y^{-q})^{2/q} - r^2 \geq (\frac{B}{2} \ln y)^{2/q} - 1 \geq (\frac{B}{4})^{2/q} (\ln y)^{2/q}.$$

Consequently,

$$K_3^{(b)}(y) \leq \int_{y^{-1/2}}^1 \frac{dr}{(B/4)^{1/q} (\ln y)^{1/q}} \leq \frac{(4/B)^{1/q}}{(\ln y)^{1/q}} \xrightarrow{y \rightarrow \infty} 0.$$

The proof is thus completed.

## B.2 Proof of Lemma 3.3

Fix  $q > 1$  and  $B > 0$ . We set

$$f_y(s) := \frac{1}{\sqrt{(1 + By^q \ln s)^{\frac{2}{q}} - s^2}}, \quad \tilde{f}_y(s) := \begin{cases} f_y(s), & \text{if } s \in ]y, 1[, \\ 0, & \text{if } s \in ]0, y]. \end{cases}$$

We have the following pointwise convergence:

$$\frac{1}{\sqrt{(1 + By^q \ln s)^{\frac{2}{q}} - s^2}} \xrightarrow{y \rightarrow 0^+} \frac{1}{\sqrt{1 - s^2}}, \quad \text{for every } s \in ]0, 1[.$$

It is our aim to prove, by Lebesgue's dominated convergence theorem, that

$$\lim_{y \rightarrow 0^+} \int_0^1 \tilde{f}_y(s) ds = \int_0^1 \frac{1}{\sqrt{1 - s^2}} ds = \frac{\pi}{2}.$$

We will show that

$$\tilde{f}_y(s) \leq g(s), \quad \text{with } g(s) := \begin{cases} \sqrt{2}, & \text{if } s \in ]0, 1/2[, \\ \frac{\sqrt{2}}{\sqrt{1 - s^2}}, & \text{if } s \in ]1/2, 1[. \end{cases} \quad (61)$$

The assertion is trivial for  $s \in ]0, y]$ . For  $s \in [y, 1/2]$  we have, for  $y$  sufficiently small,

$$(1 + By^q \ln s)^{\frac{2}{q}} - s^2 \geq (1 + By^q \ln y)^{\frac{2}{q}} - s^2 \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.$$

Hence  $f_y(s) \leq \sqrt{2}$  for every  $s \in [y, 1/2]$ , when  $y$  is sufficiently small.

Let us now consider the interval  $[1/2, 1[$ . The inequalities

$$\begin{aligned} \ln s &\geq -2(1-s), \quad \text{for every } s \in [1/2, 1], \\ (1-t)^{\frac{2}{q}} &\geq 1-Lt, \quad \text{for every } t \in [0, 1/2], \end{aligned}$$

for a suitable constant  $L > 0$ , permit us to obtain the following estimate when  $y$  is sufficiently small:

$$\begin{aligned} (1 + By^q \ln s)^{\frac{2}{q}} - s^2 &\geq (1 - 2By^q(1-s))^{\frac{2}{q}} - s^2 \\ &\geq (1 - 2LB y^q(1-s)) - s^2 \\ &= (1 - s^2) - 2LB y^q(1-s) \geq \frac{1}{2}(1 - s^2). \end{aligned}$$

Hence,

$$f_y(s) \leq \frac{\sqrt{2}}{\sqrt{1-s^2}}, \quad \text{for every } s \in [1/2, 1[,$$

when  $y$  is sufficiently small. So, the proof of the inequality in (61) is completed and the Lebesgue Theorem ensures the conclusion of the proof.

### B.3 Proof of Lemma 3.6

Set

$$I(y) := \int_1^y \frac{ds}{\sqrt{(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2}}.$$

We will need the following estimates

$$c(s-1) \leq 1 - s^{q-A} \leq d(s-1), \quad \text{for every } s \in [1, 2], \quad (62)$$

$$s^2 \leq 1 + 3(s-1), \quad \text{for every } s \in [1, 2], \quad (63)$$

$$(1+u)^{\frac{2}{q}} \geq 1 + mu, \quad \text{for every } u \in [0, 1], \quad (64)$$

for some suitable positive constants  $c, d, m$ , with  $c < d$ . We define

$$s_1(y) = 1 + \frac{1}{Bdy^A}, \quad s_2(y) = 1 + \frac{8^{q/2}}{Bcy^A}.$$

Notice that  $s_1(y) < s_2(y)$  and, if  $y$  is large enough,  $s_2(y) < 2$ . We split the integral in four parts as  $I(y) = I_1(y) + I_2(y) + I_3(y) + I_4(y)$ , where

$$\begin{aligned} I_1(y) &:= \int_1^{s_1(y)} \frac{ds}{\sqrt{(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2}}, \\ I_2(y) &:= \int_{s_1(y)}^{s_2(y)} \frac{ds}{\sqrt{(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2}}, \\ I_3(y) &:= \int_{s_2(y)}^2 \frac{ds}{\sqrt{(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2}}, \\ I_4(y) &:= \int_2^y \frac{ds}{\sqrt{(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2}}. \end{aligned}$$

*Estimate of  $I_1(y)$ .* We have, using (62)

$$By^A(1 - s^{q-A}) \leq By^A d(s-1) \leq 1, \quad \text{for every } s \in [1, s_1(y)].$$

So, we are able to use (63), (64), and then (62) to obtain, for  $y$  large enough,

$$\begin{aligned} (1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2 &\geq (1 + mBy^A(1 - s^{q-A})) - (1 + 3(s-1)) \\ &= mBy^A(1 - s^{q-A}) - 3(s-1) \\ &\geq mBy^A c(s-1) - 3(s-1) \\ &\geq (mcBy^A - 3)(s-1) \geq s-1. \end{aligned}$$

Therefore,

$$I_1(y) \leq \int_1^{s_1(y)} \frac{ds}{\sqrt{s-1}} = 2\sqrt{s_1(y)-1} \xrightarrow{y \rightarrow \infty} 0.$$

*Estimate of  $I_2(y)$ .* Using (62), (63), and (64), when  $y$  is sufficiently large and  $s \in [s_1(y), s_2(y)]$  we have

$$\begin{aligned} (1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2 &\geq (1 + By^A c(s-1))^{\frac{2}{q}} - s_2(y)^2 \\ &\geq (1 + \frac{c}{d})^{\frac{2}{q}} - s_2(y)^2 \\ &\geq 1 + m\frac{c}{d} - s_2(y)^2 \geq m\frac{c}{d} - 3\frac{8^{q/2}}{Bcy^A} \geq \frac{mc}{2d} := \eta > 0. \end{aligned}$$

Hence,

$$I_2(y) \leq \eta^{-1/2}(s_2(y) - s_1(y)) \xrightarrow{y \rightarrow \infty} 0.$$

*Estimate of  $I_3(y)$ .* For every  $s \in [s_2(y), 2]$  one has

$$\begin{aligned} (1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} &\geq (By^A(1 - s^{q-A}))^{\frac{2}{q}} \\ &\geq (By^Ac(s-1))^{\frac{2}{q}} \\ &\geq (By^Ac(s_2(y) - 1))^{\frac{2}{q}} = 8 \geq 2s^2. \end{aligned}$$

So, by (62), we get

$$(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2 \geq \frac{1}{2}(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} \geq \frac{1}{2}(By^Ac(s-1))^{\frac{2}{q}}.$$

Hence, for some  $C_1, C_2 > 0$ ,

$$I_3(y) \leq C_1 y^{-A/q} \int_{s_2(y)}^2 (s-1)^{-1/q} ds \leq C_2 y^{-A/q} \xrightarrow{y \rightarrow \infty} 0.$$

*Estimate of  $I_4(y)$ .* For every  $s \in [2, y]$ , we have  $1 - s^{q-A} \geq \mu := 1 - 2^{q-A}$ , so that, for some constant  $\sigma > 0$ , if  $y$  is sufficiently large,

$$(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2 \geq (1 + By^A\mu)^{\frac{2}{q}} - y^2 \geq 2\sigma y^{2A/q} - y^2 \geq \sigma y^{2A/q}.$$

Hence, since  $A/q > 1$ ,

$$I_4(y) \leq \frac{1}{\sqrt{\sigma}} \int_2^y \frac{1}{y^{A/q}} ds = \frac{1}{\sqrt{\sigma}} \frac{y-2}{y^{A/q}} \xrightarrow{y \rightarrow \infty} 0.$$

The proof is thus completed.

#### B.4 Proof of Lemma 3.7

Fix  $A > q > 1$  and  $B > 0$ . We set

$$f_y(s) := \frac{1}{\sqrt{(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2}}, \quad \tilde{f}_y(s) := \begin{cases} f_y(s), & \text{if } s \in ]y, 1[, \\ 0, & \text{if } s \in ]0, y]. \end{cases}$$

We have the pointwise convergence

$$\frac{1}{\sqrt{(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2}} \xrightarrow{y \rightarrow 0^+} \frac{1}{\sqrt{1 - s^2}}, \quad \text{for every } s \in ]0, 1[.$$

It is our aim to prove, by the use of the Lebesgue dominated convergence Theorem, that

$$\lim_{y \rightarrow 0^+} \int_0^1 \tilde{f}_y(s) ds = \int_0^1 \frac{1}{\sqrt{1 - s^2}} ds = \frac{\pi}{2}.$$

We will show that

$$\tilde{f}_y(s) \leq g(s), \quad \text{with } g(s) := \begin{cases} \sqrt{2}, & \text{if } s \in ]0, 1/2], \\ \frac{\sqrt{2}}{\sqrt{1-s^2}}, & \text{if } s \in ]1/2, 1[. \end{cases}$$

The assertion is trivial for  $s \in ]0, y]$ . Let us now focus our attention on the case  $s \in ]y, 1/2]$ . Since

$$0 < By^A(s^{q-A} - 1) \leq By^A(y^{q-A} - 1) = B(y^q - y^A) \leq By^q,$$

if  $y$  is sufficiently small we have

$$(1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2 \geq (1 - By^q)^{\frac{2}{q}} - s^2 \geq \frac{3}{4} - s^2 \geq \frac{1}{2},$$

and so  $f_y(s) \leq \sqrt{2}$  for every  $s \in ]y, 1/2]$ .

Concerning the interval  $[1/2, 1[$ , we have, for some positive constants  $L$  and  $M$ ,

$$\begin{aligned} s^{q-A} - 1 &\leq M(1 - s) \leq M(1 - s^2), \quad \text{for every } s \in [1/2, 1], \\ (1 + u)^{\frac{2}{q}} &\geq 1 + Lu, \quad \text{for every } u \in [-1/2, 0]. \end{aligned}$$

Then, for  $y$  sufficiently small,

$$\begin{aligned} (1 + By^A(1 - s^{q-A}))^{\frac{2}{q}} - s^2 &= (1 - By^A(s^{q-A} - 1))^{\frac{2}{q}} - s^2 \\ &\geq 1 - LBy^A(s^{q-A} - 1) - s^2 \\ &\geq 1 - LMB y^A(1 - s^2) - s^2 \\ &\geq (1 - s^2)(1 - LMB y^A) \geq \frac{1}{2}(1 - s^2). \end{aligned}$$

So,

$$f_y(s) \leq \frac{\sqrt{2}}{\sqrt{1-s^2}}, \quad \text{for every } s \in [1/2, 1[.$$

Hence, the Lebesgue Theorem can be applied, thus concluding the proof.

## B.5 Proof of Lemma 3.11

We fix  $N > e^{1/A}$  such that

$$\eta := 1 - N^{A-q} > 1/q. \tag{65}$$

We split the integral

$$I(y) := \int_1^y \frac{ds}{\sqrt{(1 + qy^A(s^{q-A} - 1))^{2/q} - s^2}}$$

as  $I(y) = I_1(y) + I_2(y)$  where

$$I_1(y) := \int_1^N \frac{ds}{\sqrt{(1 + qy^A(s^{q-A} - 1))^{2/q} - s^2}},$$

$$I_2(y) := \int_N^y \frac{ds}{\sqrt{(1 + qy^A(s^{q-A} - 1))^{2/q} - s^2}}.$$

We first prove that

$$\lim_{y \rightarrow +\infty} I_1(y) = 0.$$

Notice that the function  $h(s; y) = (1 + qy^A(s^{q-A} - 1))^{1/q}$  is such that

$$\frac{\partial^2 h}{\partial s^2}(s; y) = u(s; y) \left[ (q - A - 1)(1 - qy^A) - Ay^A s^{q-A} \right],$$

$$\text{with } u(s; y) = y^A(q - A) s^{q-A-2} \left[ 1 + qy^A(s^{q-A} - 1) \right]^{\frac{1}{q}-2} > 0.$$

Setting, for  $y$  large so to have  $qy^A > 1$ ,

$$s_0(y) := \begin{cases} 1, & \text{if } q - A > 1, \\ \max \left\{ 1, \left( \frac{(q-A-1)(1-xy^A)}{Ay^A} \right)^{\frac{1}{q-A}} \right\}, & \text{if } q - A \leq 1, \end{cases}$$

we see that  $s_0(y) \in [1, N[$ , since  $N > e^{1/A}$ , and the function  $h(\cdot; y)$  is convex on  $[1, s_0(y)]$  and concave on  $[s_0(y), N]$ . Hence  $h(s; y) \geq 1 + \mu(y)(s - 1)$ , where

$$\mu(y) = \min \left\{ \frac{\partial h}{\partial s}(1; y), \frac{h(N; y) - h(1; y)}{N - 1} \right\} = \min \left\{ (q - A)y^A, \frac{h(N; y) - 1}{N - 1} \right\}.$$

Since  $\mu(y) \rightarrow +\infty$  as  $y \rightarrow +\infty$ , for  $y$  large enough we have

$$\begin{aligned} (1 + qy^A(s^{q-A} - 1))^{2/q} - s^2 &\geq 2 \left[ (1 + qy^A(s^{q-A} - 1))^{1/q} - s \right] \\ &\geq 2[1 + \mu(y)(s - 1) - s] \\ &= 2(\mu(y) - 1)(s - 1) \geq \mu(y)(s - 1), \end{aligned}$$

and then

$$I_1(y) \leq \frac{1}{\sqrt{\mu(y)}} \int_1^N \frac{ds}{\sqrt{s-1}} \xrightarrow{y \rightarrow \infty} 0.$$

We now consider the integral  $I_2(y)$ . Setting  $s = yt$ , we get

$$I_2(y) = \int_{N/y}^1 \frac{dt}{\sqrt{D_y(t)}},$$

where

$$D_y(t) := \frac{(1 + qy^A((yt)^{q-A} - 1))^{2/q}}{y^2} - t^2 = (y^{-q} + q(t^{q-A} - y^{A-q}))^{2/q} - t^2.$$

We notice that, as  $y \rightarrow +\infty$ , the function

$$f_y(t) = \begin{cases} 0, & \text{if } t \in ]0, N/y], \\ D_y(t)^{-1/2}, & \text{if } t \in ]N/y, 1[, \end{cases}$$

converges pointwise in  $]0, 1[$  to the function

$$f(t) = D(t)^{-1/2}, \quad \text{where} \quad D(t) := q^{2/q} t^{2(q-A)/q} - t^2 = t^{2(q-A)/q} (q^{2/q} - t^{2A/q}).$$

We will prove, by the Lebesgue Theorem and the change of variable  $t^{A/q} = \xi$ , that

$$\lim_{y \rightarrow +\infty} I_2(y) = \lim_{y \rightarrow +\infty} \int_0^1 f_y(t) dt = \int_0^1 f(t) dt = \frac{q}{A} \int_0^1 \frac{d\xi}{\sqrt{q^{2/q} - \xi^2}} = \frac{q}{A} \arcsin(q^{-1/q}).$$

The remaining part of the proof is devoted to find a dominating integrable function  $g$  such that  $f_y(t) \leq g(t)$  for every  $t \in ]0, 1[$ .

We focus our attention on  $t \in [N/y, 1[$ . Since  $yt \geq N$ , recalling (65) we can deduce that

$$(yt)^{q-A} - 1 \geq \eta(yt)^{q-A},$$

and so

$$\begin{aligned} D_y(t) &= \frac{[1 + qy^A((yt)^{q-A} - 1)]^{2/q}}{y^2} - t^2 \geq \frac{[qy^A \eta(yt)^{q-A}]^{2/q}}{y^2} - t^2 \\ &= (q\eta)^{2/q} t^{2(q-A)/q} - t^2. \end{aligned}$$

Since  $t < 1$  and  $2(q-A)/q < 2$ , one has  $t^2 \leq t^{2(q-A)/q}$ . Hence, for  $y$  large enough,

$$D_y(t) \geq [(q\eta)^{2/q} - 1] t^{2(q-A)/q}, \quad \text{for every } t \in [N/y, 1[,$$

and since  $q\eta > 1$  by construction, there is a positive constant  $C$  such that, for  $y$  sufficiently large,

$$f_y(t) \leq C t^{-(q-A)/q}, \quad \text{for every } t \in [N/y, 1[.$$

We have found the dominating function  $g(t) = t^{(A-q)/q}$ , thus completing the proof.

## B.6 Proof of Lemma 3.12

We write

$$I(y) := \int_y^1 \frac{ds}{\sqrt{\mathcal{D}_y(s)}}, \quad \text{with} \quad \mathcal{D}_y(s) = \left(1 - qy^A(1 - s^{q-A})\right)^{2/q} - s^2.$$

Setting

$$f_y(s) = \begin{cases} 0, & \text{if } s \in ]y_0, y], \\ [\mathcal{D}_y(s)]^{-1/2}, & \text{if } s \in ]y, 1[, \end{cases}$$

since  $qy^A \rightarrow 1$  as  $y \rightarrow y_0^+$ , we see that, pointwise on  $]y_0, 1[$ ,

$$f_y(s) \rightarrow f(s) = \mathcal{D}(s)^{-1/2}, \quad \text{with} \quad \mathcal{D}(s) := s^{2(q-A)/q} - s^2.$$

We will prove, by the Lebesgue Theorem and the change of variable  $s^{A/q} = \xi$ , that

$$\lim_{y \rightarrow y_0^+} I(y) = \lim_{y \rightarrow y_0^+} \int_0^1 f_y(s) ds = \int_0^1 f(s) ds = \frac{q}{A} \int_{q^{-1/q}}^1 \frac{d\xi}{\sqrt{1 - \xi^2}} = \frac{q}{A} \left( \frac{\pi}{2} - \arcsin(q^{-1/q}) \right).$$

The remaining part of the proof is devoted to find a dominating integrable function  $g$  such that  $f_y(s) \leq g(s)$  for every  $s \in ]y_0, 1[$ .

The function

$$s \mapsto h(s; y) = \left(1 - qy^A(1 - s^{q-A})\right)^{1/q}$$

is increasing,  $h(1; y) = 1$  and  $\frac{\partial h}{\partial s}(1; y) = (qy^A)^{\frac{q-A}{q}} < 1$  for  $(y - y_0)$  small enough. So, we can deduce the existence of  $\eta < 1$  and  $s_0 \in ]y_0, 1[$  sufficiently near 1, such that, for  $(y - y_0)$  small,

$$h(s; y) > 1 + \eta(s - 1), \quad \text{for every } s \in [s_0, 1].$$

Hence,

$$\begin{aligned} \left(1 - qy^A(1 - s^{q-A})\right)^{2/q} - s^2 &\geq \left(1 - qy^A(1 - s^{q-A})\right)^{1/q} - s \\ &\geq 1 + \eta(s - 1) - s = (1 - s)(1 - \eta), \end{aligned}$$

so that  $f_y(s) \leq g(s) := [(1 - s)(1 - \eta)]^{-1/2}$  for every  $s \in ]s_0, 1[$ .

Let us now focus our attention on the case  $s \in [y_0, s_0]$ . The function

$$k(s; \zeta) := [(1 + \zeta)s^{q-A} - \zeta]^{2/q} - s^2$$

is such that  $k(s; 0) = \mathcal{D}(s) \geq 2\delta$  for every  $s \in [y_0, s_0]$ , for a suitable  $\delta > 0$ . Then, there exists  $\zeta_0 > 0$  such that, if  $|\zeta| < \zeta_0$ , then  $k(s; \zeta) \geq \delta$  for every  $s \in [y_0, s_0]$ . As a consequence, taking  $\zeta = qy^A - 1$ , for  $(y - y_0)$  small we have  $qy^A < 1 + \zeta_0$ , and so

$$\mathcal{D}_y(s) = \left((qy^A)s^{q-A} - (qy^A - 1)\right)^{2/q} - s^2 = k(s; qy^A - 1) \geq \delta,$$

thus giving us  $f_y(s) \leq g(s) := \delta^{-1/2}$  for every  $s \in [y_0, s_0]$ . The proof is completed.

**Acknowledgement.** We warmly thank Natnael Gezahegn Mamo for friendly discussions. Both authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The research contained in this paper was carried out within the framework of DEG1 – Differential Equations Group Of North-East.

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Mathematics Subject Classification: 34C25.

Keywords: Bertrand's theorem; periodic solutions; Newtonian systems; p-Laplacian.