Multiple periodic solutions of nonlinear oscillators with velocity dependent stiffness

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To our friend Jean Mawhin, with gratitude

Abstract

By the use of the Poincaré–Birkhoff theorem, we generalize several results on the existence and multiplicity of periodic solutions for scalar second order differential equations with a periodic nonlinearity, the paradigm of which is the pendulum equation. As a particular case, we extend a recent result by Amster, Cid, and Mawhin on a relativistic model of the pendulum.

1 Introduction

The periodically forced pendulum equation has attracted the attention of several mathematicians for more than a century (see, e.g., [19] for a review). Let us mention a few fundamental results concerning the classical equation

$$x'' + \alpha \sin x = f(t)\,.$$

In 1922, Hamel [16] proved that if $f(t)$ is T-periodic and has zero mean, i.e.,

$$\frac{1}{T}\int_0^T f(t)\,dt = 0\,, \tag{1}$$

then there exists a T-periodic solution. The proof is variational, by minimization of the action functional. This result has been rediscovered in [8, 31], and later extended by Mawhin and Willem in [22], proving that under the above assumption (1) there exist indeed at least two geometrically distinct T-periodic solutions. The first solution is obtained by minimizing the action functional, the second one by a mountain pass procedure. Different proofs have been proposed in [11, 14].

After the pioneering paper [4] on differential equations involving the relativistic differential operator, the existence of a T-periodic solution for the so-called relativistic pendulum,

$$\left(\frac{x'}{\sqrt{1-|x'|^2}}\right)' + \alpha \sin x = f(t)\,,$$

was first provided by Brezis and Mawhin in [6], under assumption (1). Later on, Bereanu and Torres [5] showed that, also for this equation, there are indeed at least two geometrically distinct T-periodic solutions. The proof is somewhat similar to the one in [22]. See also [7] for an updated survey on this type of equations.

In a recent paper by Amster, Cid, and Mawhin [2], the multiplicity of T-periodic solutions was proved for the equation

$$\left(\frac{x'}{\sqrt{1-|x'|^2}}\right)' + \frac{\beta}{\sqrt{1-|x'|^2}} \sin x = f(t)\,, \tag{2}$$

under the same assumption (1) on the forcing term $f(t)$. At first sight the presence of the derivative dependent term seems to exclude the possibility of using variational methods in this case. Surprisingly enough, however, an integrating factor permits to discover the Lagrangian nature of the equation, and the proof is carried out by a Lusternik–Schnirelmann type argument developed in [24, 29].

Let us mention that the study of equation (2) was motivated by a model of the relativistic pendulum with relativistic mass effects, as proposed in [1, 9, 15]. Notice further that the results in [2, 5, 22] were verified also when $\sin x$ is replaced by any continuous 2π-periodic function $g(x)$, provided that $\int_0^{2\pi} g(x)\,dx = 0$, thus guaranteeing that the function $G(x) = \int_0^x g(s)\,ds$ is 2π-periodic.

We will extend the above results to more general equations of the type

$$(\phi(x'))' + \xi(x')g(x) = f(t)\,. \tag{3}$$

A structural assumption relating ϕ and ξ will permit us to write an equivalent Hamiltonian system

$$x' = \partial_y H(t,x,y)\,, \quad y' = -\partial_x H(t,x,y)\,, \tag{4}$$

where $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, T-periodic in t, with continuous partial derivatives $\partial_x H(t,x,y)$ and $\partial_y H(t,x,y)$. Assumption (1) will guarantee that if $(x(t), y(t))$ is a T-periodic solution of (4), then $x(t)$ is a T-periodic solution of (3).

We will then apply a rather recent version of the Poincaré–Birkhoff theorem, first proved in [13], which we now recall, for the reader's convenience.

Theorem 1. *Assume in addition $H(t, x, y)$ to be 2π-periodic with respect to x. Let $a, b : \mathbb{R} \to \mathbb{R}$ be two continuous and 2π-periodic functions such that $a(x) < b(x)$ for every x, and define*

$$\mathcal{G}_a = \big\{(x, y) \in \mathbb{R}^2 : y = a(x)\big\}\,, \qquad \mathcal{G}_b = \big\{(x, y) \in \mathbb{R}^2 : y = b(x)\big\}\,,$$

and

$$\mathcal{S} = \big\{(x, y) \in \mathbb{R}^2 : a(x) \le y \le b(x)\big\}\,.$$

Let all the solutions $z = (x, y)$ of (4) starting with $z(0) \in \mathcal{S}$ be defined on $[0, T]$ and assume that

$$\begin{cases} z(0) \in \mathcal{G}_a & \Rightarrow \quad x(T) - x(0) < 0\,, \\ z(0) \in \mathcal{G}_b & \Rightarrow \quad x(T) - x(0) > 0\,. \end{cases} \tag{5}$$

Then, system (4) has at least two geometrically distinct T-periodic solutions $z = (x, y)$, with $z(0) \in \mathcal{S}$.

It is time to specify what we mean by *geometrically distinct* solutions: these are solutions $(x(t), y(t))$ which cannot be obtained from each other by just adding an integer multiple of 2π to $x(t)$.

Assumption (5) is usually called a *twist condition*, and the same conclusion also holds if the inequalities in (5) are reversed. Notice that in Theorem 1 no uniqueness assumption is made for the solutions of initial value problems.

By the use of Theorem 1 we will obtain the multiplicity of periodic solutions for an equation like

$$\left(\frac{x'}{\sqrt{1 - |x'|^2}}\right)' + \left(\alpha + \frac{\beta}{\sqrt{1 - |x'|^2}}\right) g(x) = f(t)\,, \tag{6}$$

with $g(x)$ as above, thus unifying the results in [2] and [5]. Moreover, a variant of our theorem can be applied to equations of the type

$$(|x'|^{p-2}x')' + (\alpha + \beta|x'|^p)\, g(x) = f(t)\,, \tag{7}$$

where $p > 1$. The case $\beta = 0$ has been treated by Mawhin in [20]. There seems to be no contribution in the literature on this type of equations when $p \neq 2$ and $\beta \neq 0$. Some results are available, for the autonomous equation, in the case $p = 2$ (see, e.g., [3, 9, 17, 25]).

The paper is organized as follows. In Section 2 we state our main results for the existence of two periodic solutions, in the general case. Corollaries for the particular equations mentioned above are also provided. The proofs are carried out in Section 3.

In Section 4 we consider a perturbative problem, so to discover the existence of multiple subharmonic solutions. The idea is to exploit the twist property of the solutions of the autonomous system emanating from the origin, assuming $g(0) = 0$. A detailed analysis when $g(x) = \sin x$ is also carried out.

In Section 5 we obtain the existence and multiplicity of the so-called *periodic solutions of the second kind*, also known as *running solutions*. In the case of the pendulum equation, they correspond to solutions which can make several rotations around the fulcrum.

Finally, in Section 6 we propose some open problems and suggest further possible developments of the theory.

2 The main results

We are interested in finding T-periodic solutions of equation (3). We assume $\phi :]-\rho, \rho[\to \mathbb{R}$ to be an increasing homeomorphism, for some $\rho \in]0, +\infty]$, with $\phi(0) = 0$, while $\xi :]-\rho, \rho[\to [0, +\infty[$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions. The function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be locally integrable and T-periodic. We say that $x : \mathbb{R} \to \mathbb{R}$ is a *solution* of (3) if it is continuously differentiable, the function $t \mapsto \phi(x'(t))$ is absolutely continuous, and the equation is satisfied almost everywhere.

We will distinguish two cases: $\rho < +\infty$ (the singular case) and $\rho = +\infty$ (the regular case, when $]-\rho, \rho[= \mathbb{R}$). Here is our first result, concerning the singular case.

Theorem 2. *Assume $g : \mathbb{R} \to \mathbb{R}$ to be 2π-periodic, with $\int_0^{2\pi} g(x)\,dx = 0$. If $\rho < +\infty$ and*

$$\xi(v) = \xi(0) + v\phi(v) - \int_0^v \phi(\nu)\,d\nu\,, \qquad \textit{for every } v \in]-\rho, \rho[\,, \tag{8}$$

then there are at least two geometrically distinct T-periodic solutions of (3)*, provided that f satisfies* (1)*.*

The proof will be carried out in Section 3. As a consequence of the above theorem, we have the following.

Corollary 3. *Assume $g : \mathbb{R} \to \mathbb{R}$ to be 2π-periodic, with $\int_0^{2\pi} g(x)\,dx = 0$. Then, for any constants α and β, equation* (6) *has at least two geometrically distinct T-periodic solutions, provided that f satisfies* (1)*.*

Proof. The case $\beta = 0$ follows from [5]. Assume now $\beta \neq 0$, and define

$$\phi(v) = \frac{v}{\sqrt{1 - v^2}}\,, \quad \xi(v) = \frac{\alpha}{\beta} + \frac{1}{\sqrt{1 - v^2}}\,. \tag{9}$$

Then Theorem 2 applies, with $g(x)$ replaced by $\beta g(x)$. □

The above corollary thus extends the main result in [5], where the case $\beta = 0$ was analyzed. It also generalizes the result in [2], where the case $\alpha = 0$ was treated.

Assume now $\rho = +\infty$. Here is the corresponding result.

Theorem 4. *Assume $g : \mathbb{R} \to \mathbb{R}$ to be 2π-periodic, with $\int_0^{2\pi} g(x)\,dx = 0$, let $\phi, \xi : \mathbb{R} \to \mathbb{R}$ be such that* (8) *holds. Assume moreover*

$$\liminf_{|v|\to\infty} \frac{\xi(v)}{|v|} > 0\,. \tag{10}$$

Then, there are at least two geometrically distinct T-periodic solutions of (3)*, provided that f satisfies* (1)*.*

The proof of the theorem is postponed to Section 3. As a consequence, we have the following.

Corollary 5. *Assume $g : \mathbb{R} \to \mathbb{R}$ to be 2π-periodic, with $\int_0^{2\pi} g(x)\,dx = 0$. Then, for any constants α and β, equation* (7) *has at least two geometrically distinct T-periodic solutions, for any $p > 1$, provided that f satisfies* (1)*.*

Proof. If $\beta = 0$, the result follows from [11], after noticing that Theorem 1 does not require the uniqueness for the solutions of initial value problems (hence [11, Assumption (2.4)] is not needed). If $\beta \neq 0$, defining

$$\phi(v) = |v|^{p-2}v\,, \quad \xi(v) = \frac{p-1}{p}\Big(\frac{\alpha}{\beta} + |v|^p\Big)\,, \tag{11}$$

Theorem 4 applies with $g(x)$ replaced by $\frac{p}{p-1}\beta g(x)$. □

Here is the particular case $p = 2$ and $g(x) = \sin x$.

Corollary 6. *For any constants α and β, the equation*

$$x'' + (\alpha + \beta(x')^2)\sin x = f(t) \tag{12}$$

has at least two geometrically distinct T-periodic solutions, provided that f satisfies (1)*.*

We have thus also extended [22], where the result for the case $\beta = 0$ was first obtained.

Remark 7. *In the autonomous case (i.e., when $f = 0$), it can be seen that equation [2, Eq. (5)], i.e.,*

$$\left(\frac{\ell\, x'(t)}{\sqrt{1 - \frac{\ell^2 (x')^2}{c^2}}}\right)' + \frac{g}{\sqrt{1 - \frac{\ell^2 (x')^2}{c^2}}}\sin x = 0\,,$$

where g denotes the constant gravitational acceleration, is equivalent to the following:

$$x'' + \Big(\frac{g}{\ell} - \frac{g\ell}{c^2}(x')^2\Big)\sin x = 0\,.$$

This equation was proposed in [9, Eq. (9)] as a model of the relativistic pendulum with a relativistic correction for the mass.

3 Proofs of Theorems 2 and 4

In this section we provide a proof of both Theorems 2 and 4 by the use of Theorem 1. Define

$$F(t) = \int_0^t f(\tau)\,d\tau\,, \qquad \eta(w) = \xi(0) + \int_0^w \phi^{-1}(u)\,du\,,$$

and let $G(x)$ be any primitive of $g(x)$, e.g., $G(x) = \int_0^x g(s)\,ds$. Notice that, by assumption (8),

$$\eta(w) = \xi\big(\phi^{-1}(w)\big)\,, \qquad \text{for every } w \in \mathbb{R}\,. \tag{13}$$

We first need to prove the following lemmas.

Lemma 8. *Equation* (3) *is equivalent to the planar Hamiltonian system*

$$\begin{cases} x' = \phi^{-1}(e^{-G(x)}y + F(t)) \\ y' = g(x)\big[y\,\phi^{-1}(e^{-G(x)}y + F(t)) - e^{G(x)}\eta(e^{-G(x)}y + F(t))\big]\,, \end{cases} \tag{14}$$

whose Hamiltonian function $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is defined as

$$H(t,x,y) = e^{G(x)}\eta(e^{-G(x)}y + F(t))\,.$$

Proof. Since η is a primitive of ϕ^{-1}, a direct computation shows that (14) is indeed the system associated with the Hamiltonian function $H(t,x,y)$.

Now, let (x,y) be a solution of (14). Notice that the first equation of the system can be written as

$$y = e^{G(x)}(\phi(x') - F(t))\,, \tag{15}$$

from which we get

$$y' = e^{G(x)}\big[g(x)x'(\phi(x') - F(t)) + (\phi(x'))' - f(t)\big]\,,$$

almost everywhere. Equating with the second equation in (14), after dividing by $e^{G(x)}$, we have

$$g(x)x'(\phi(x') - F(t)) + (\phi(x'))' - f(t) = g(x)\big[ye^{-G(x)}\phi^{-1}(w) - \eta(w)\big]\,,$$

where $w = e^{-G(x)}y + F(t)$. Then, using (15) and simplifying, we have that

$$(\phi(x'))' - f(t) = -g(x)\eta(\phi(x'))\,,$$

almost everywhere, and by (13) we see that x is a solution of (3).

Vice versa, if x is a solution of (3), setting $y = e^{G(x)}(\phi(x') - F(t))$, the same computations show that (x, y) solves (14). □

Notice that the Hamiltonian function $H(t, x, y)$ is T-periodic in t and 2π-periodic in x. By (1), it is easily seen that if $(x(t), y(t))$ is a T-periodic solution of (14), then $x(t)$ is a T-periodic solution of (3).

Lemma 9. *The solutions of* (14) *with any initial value* $(x(0), y(0))$ *are globally defined on* $\mathbb{R}$.

Proof. If $\rho < +\infty$, then $\phi^{-1} : \mathbb{R} \to]-\rho, \rho[$ is bounded, and since $G(x)$ is 2π-periodic, hence bounded, the right-hand side of (14) has an at most linear growth in (x, y), and the conclusion follows.

Let us consider now the case $\rho = +\infty$. By (10) there exist $\delta > 0$ and $\gamma \geq 0$ such that

$$\xi(s) \geq \delta|s| - \gamma\,, \qquad \text{for every } s \in \mathbb{R}\,. \tag{16}$$

Let $(x(t), y(t))$ be a solution of (14), and define

$$w(t) = e^{-G(x(t))}y(t) + F(t)\,, \qquad u(t) = e^{G(x(t))}(\eta(w(t)) + \gamma)\,.$$

After noticing that $u(t) = H(t, x(t), y(t)) + \gamma e^{G(x(t))}$, we compute

$$\begin{aligned} u'(t) &= \frac{\partial H}{\partial t}(t, x(t), y(t)) + \gamma e^{G(x(t))}g(x(t))x'(t) \\ &= e^{G(x(t))}\phi^{-1}(w(t))\big(f(t) + \gamma g(x(t))\big)\,, \end{aligned}$$

for almost every t. Since, by (16),

$$|\phi^{-1}(w)| \leq \frac{1}{\delta}(\xi(\phi^{-1}(w)) + \gamma) = \frac{1}{\delta}(\eta(w) + \gamma)\,,$$

we obtain

$$|u'(t)| \leq \frac{|f(t)| + \gamma\|g\|_\infty}{\delta}\, u(t)\,.$$

Hence, $u(t)$ is bounded on bounded intervals. Therefore, since $G(x)$ is 2π-periodic, hence bounded, we deduce that also

$$\xi(x'(t)) = \eta(w(t)) = e^{-G(x(t))}u(t) - \gamma$$

is bounded on bounded intervals, and by (16), the same is true for $x'(t)$. Using (15), the conclusion easily follows. □

Here is the last lemma.

Lemma 10. *For every* $r > 0$ *there is an* $R > r$ *with the property that, if* (x, y) *is a solution of* (14) *such that* $|y(0)| \geq R$, *then* $|y(t)| \geq r$ *for every* $t \in [0, T]$.

Proof. It is a standard consequence of the fact that the solutions of (14) are globally defined, as stated in Lemma 9, and the Hamiltonian function $H(t,x,y)$ is 2π-periodic in x. □

We are now able to finish the proof. By Lemma 10, if $y(0)$ is chosen to be positive and large enough, then

$$e^{-G(x(t))}y(t)+F(t)>0\,,\qquad \text{for every } t\in[0,T]\,,$$

implying that $x'(t)>0$ for every $t\in[0,T]$, whence $x(T)-x(0)>0$. Similarly, if $y(0)$ is chosen to be negative and large enough, then $x(T)-x(0)<0$. The twist condition (5) is thus verified, taking $a(x)$ and $b(x)$ to be constant functions, with $a>0$ large enough and $b=-a$. Theorem 1 can then be applied, thus concluding the proofs of Theorems 2 and 4.

4 Perturbative results

We use the notation $\|f\|_1=\int_0^T|f(t)|\,dt$, and study equation (3) when $\|f\|_1$ is sufficiently small.

Let us write the autonomous equation

$$(\phi(x'))'+\xi(x')g(x)=0\,, \tag{17}$$

with the associated autonomous Hamiltonian system

$$\begin{cases} x'=\phi^{-1}(e^{-G(x)}y)\\ y'=g(x)\big[y\,\phi^{-1}(e^{-G(x)}y)-e^{G(x)}\eta(e^{-G(x)}y)\big]\,, \end{cases} \tag{18}$$

whose Hamiltonian function is

$$H_0(x,y)=e^{G(x)}\eta(e^{-G(x)}y)\,.$$

We now assume $\phi:\,]-\rho,\rho[\,\to\mathbb{R}$ to be an increasing *diffeomorphism*, for some $\rho\in\,]0,+\infty]$, with $\phi(0)=0$, while $\xi:\,]-\rho,\rho[\,\to[0,+\infty[$ and $g:\mathbb{R}\to\mathbb{R}$ are continuously differentiable functions. Notice that, in this case, assumption (8) is equivalent to

$$\xi'(v)=v\phi'(v)\,,\qquad \text{for every } v\in\,]-\rho,\rho[\,. \tag{19}$$

We now first provide the results in the case of a general function $g(x)$, and later concentrate on the specific case $g(x)=\sin x$.

4.1 The general $g(x)$ situation

Here is our result.

Theorem 11. *Assume* (19) *and let* g *be* 2π*-periodic, with* $g(0) = 0$ *and* $\xi(0)g'(0) > 0$*. Let* m *be an integer such that*

$$mT > \frac{2\pi\,\phi'(0)}{\sqrt{\xi(0)g'(0)}}\,. \tag{20}$$

Then, there is an $\varepsilon_m > 0$ *with the following property: if* $\|f\|_1 \le \varepsilon_m$ *and* (1) *holds, equation* (3) *has at least two periodic solutions* $x_1(t)$*,* $x_2(t)$ *having minimal period* mT*. Moreover, for* $j = 1, 2$*, the curves* $t \mapsto (x_j(t), x_j'(t))$ *do not cross the origin, and rotate around it exactly once in the time interval* $[0, mT]$*.*

Proof. Let us just sketch the idea of the proof, which follows the argument in [12, Corollary 3.5]. The Hamiltonian function H_0 is twice continuously differentiable. Besides the origin, other equilibria of (18) arise at $(x, 0)$, when $g(x) = 0$. Hence, by the 2π-periodicity of g, there surely exist infinitely many equilibria.

Since $\phi'(0) > 0$ and $\xi(0)g'(0) > 0$, the origin is a non-degenerate local minimum of the Hamiltonian function. We notice that the solutions of the linearized equation are periodic with minimal period $2\pi\phi'(0)/\sqrt{\xi(0)g'(0)}$. The local minimum $(0, 0)$ of the Hamiltonian function then generates a period annulus for system (18) which does not cover the whole plane, because of the presence of other equilibria. This fact forces the periodic solutions of (18) to have a period which approaches infinity as the orbits grow in their dimension and get near to another equilibrium. Passing to suitable action-angle coordinates in system (18) (see [10]), the solutions near the origin rotate more than once in the time interval $[0, mT]$, while those with sufficiently large amplitude cannot make a complete rotation in the same time interval. This dynamics generates the twist condition in Theorem 1, with T replaced by mT. The twist is then preserved when introducing a forcing term $f(t)$ with $\|f\|_1$ sufficiently small, i.e., $\|F\|_\infty$ small enough. Hence, Theorem 1 applies, providing two mT-periodic solutions $(x_j(t), y_j(t))$ of system (14), with $j = 1, 2$, which rotate exactly once around the origin in the time interval $[0, mT]$.

Since ϕ is a homeomorphism, it can be easily seen that the curves $t \mapsto (x_j(t), x_j'(t))$ have the same property. This fact also guarantees that mT is the minimal period, and the proof is thus completed. □

We emphasize that in Theorem 11 the function $G(x)$ needs not be 2π-periodic. Let us state the following corollaries.

Corollary 12. *Assume g to be 2π-periodic and differentiable, with $g(0) = 0$. Let α and β be real constants, with $(\alpha + \beta)g'(0) > 0$, and let m be an integer for which*

$$mT > \frac{2\pi}{\sqrt{(\alpha + \beta)g'(0)}}\,.$$

Then, the conclusion of Theorem 11 holds for equation (6).

Proof. It follows from Theorem 11 taking the functions $\phi(v)$, $\xi(v)$ as in (9), and $g(x)$ replaced by $\beta g(x)$. □

Corollary 13. *Assume g to be 2π-periodic and differentiable, with $g(0) = 0$. Let α and β be real constants, with $\alpha g'(0) > 0$, and let m be an integer for which*

$$mT > \frac{2\pi}{\sqrt{\alpha g'(0)}}\,.$$

Then, the conclusion of Theorem 11 holds for equation (7) *with $p = 2$.*

Proof. Take

$$\phi(v) = v\,, \quad \xi(v) = \frac{1}{2}\Big(\frac{\alpha}{\beta} + v^2\Big)\,, \tag{21}$$

and apply Theorem 11 with $g(x)$ replaced by $2\beta g(x)$. □

Here is the corresponding result for equation (12).

Corollary 14. *Let α and β be real constants, with $\alpha > 0$, and let m be an integer for which*

$$mT > \frac{2\pi}{\sqrt{\alpha}}\,.$$

Then, the conclusion of Theorem 11 holds for equation (12).

Notice that, if $1 < p < 2$ in equation (7), assumption (20) could never be satisfied, since in this case $\phi'(0) = +\infty$. On the other hand, if $p > 2$, the approach used in [12] cannot be applied since the associated Hamiltonian function fails to be twice continuously differentiable with respect to the phase variables.

4.2 The particular case $g(x) = \sin x$

When $g(x) = \sin x$ we can be more precise about the behavior of the solutions for the perturbed equations (6) and (7). Let us first consider the autonomous equation

$$\left(\frac{x'}{\sqrt{1 - |x'|^2}}\right)' + \left(\alpha + \frac{\beta}{\sqrt{1 - |x'|^2}}\right)\sin x = 0\,, \tag{22}$$

which comes from the Hamiltonian function

$$H_1(x,y) = \begin{cases} e^{-\beta\cos x}\Big(\dfrac{\alpha}{\beta} + \sqrt{1+e^{2\beta\cos x}y^2}\Big)\,, & \beta \neq 0\,, \\ \sqrt{1+y^2} - \alpha\cos x\,, & \beta = 0\,. \end{cases} \tag{23}$$

Notice that, when $\beta \neq 0$, referring to Lemma 8, we have chosen $g(x) = \beta\sin x$ and $G(x) = -\beta\cos x$, while $\phi(v)$ and $\xi(v)$ are defined as in (9) so that, by (13), we have $\eta(w) = \alpha/\beta + \sqrt{1+w^2}$. The reader can visualize in Figure 1 the level lines of H_1 when $\alpha = 1$ and $\beta = 1$.

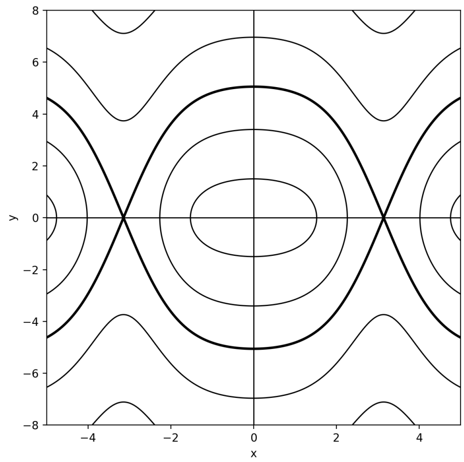

Figure 1: *The level lines of* (23) *with* $\alpha = \beta = 1$.

In the estimates below, when writing $f(t) \sim g(t)$ as $t \to t_0$ we mean that $\lim_{t\to t_0} f(t)/g(t) = 1$.

Proposition 15. *Assume $\alpha + \beta > 0$. Then the equilibria $((2k+1)\pi, 0)$, with $k \in \mathbb{Z}$, are hyperbolic saddles. Moreover, the critical energy level through $(\pi, 0)$ contains two invariant curves $\Gamma^\pm$ connecting $(-\pi, 0)$ and $(\pi, 0)$ in opposite time directions, symmetrically, given on $[-\pi, \pi]$ by the graphs $y = \pm Y(x)$, where*

$$Y(x)^2 = \begin{cases} e^{-2\beta\cos x}\Big[\Big(\Big(\dfrac{\alpha}{\beta}+1\Big)e^{\beta(1+\cos x)} - \dfrac{\alpha}{\beta}\Big)^2 - 1\Big]\,, & \beta \neq 0, \\ \big(1+\alpha(1+\cos x)\big)^2 - 1\,. & \beta = 0. \end{cases} \tag{24}$$

These curves are heteroclinic orbits: if $(x(t), y(t))$ is a solution on Γ^+, then

$$x(t) + \pi \sim C\, e^{-\sqrt{\alpha+\beta}\,|t|} \quad \textit{as} \;\; t \to -\infty\,,$$

$$\pi - x(t) \sim C\, e^{-\sqrt{\alpha+\beta}\,t} \quad \textit{as} \;\; t \to +\infty\,,$$

for some constant $C > 0$. Similarly on Γ^-.

Proof. It is easy to see that the equilibria are precisely $(k\pi, 0)$, with $k \in \mathbb{Z}$. To determine the type of $(\pi, 0)$, we use the scalar equation (22). Linearizing at $(x, x') = (\pi, 0)$ we obtain the equation $\eta'' - (\alpha+\beta)\eta = 0$, showing that $(\pi, 0)$ is a hyperbolic saddle if and only if $\alpha + \beta > 0$.

For $\beta = 0$ the critical level through $(\pi, 0)$ is

$$H_1(x, y) = H_1(\pi, 0) \quad \Leftrightarrow \quad \sqrt{1+y^2} - \alpha \cos x = \alpha + 1\,,$$

which gives two invariant graphs $y = \pm Y(x)$ joining $(-\pi, 0)$ and $(\pi, 0)$.

Let us now focus on the case $\beta \neq 0$ and consider the critical level set $H_1(x, y) = H_1(\pi, 0)$, namely

$$e^{-\beta \cos x}\Big(\frac{\alpha}{\beta} + \sqrt{1 + e^{2\beta \cos x} y^2}\Big) = e^{\beta}\Big(\frac{\alpha}{\beta} + 1\Big)\,,$$

which is equivalent to

$$\sqrt{1 + e^{2\beta \cos x} y^2} = \Big(\frac{\alpha}{\beta} + 1\Big) e^{\beta(1+\cos x)} - \frac{\alpha}{\beta}\,.$$

Then, for $x \in]-\pi, \pi[$ one can easily see that this equation defines the two smooth branches $y = \pm Y(x)$ joining $(-\pi, 0)$ to $(\pi, 0)$, as in the statement. Indeed, the assumption $\alpha+\beta > 0$ guarantees the coherence of all the formulas. The final estimate in the statement follows from the hyperbolicity of these two equilibria, and from the linearized equation $\eta'' - (\alpha + \beta)\eta = 0$, where $\eta(t) = x(t) + \pi$ or $\eta(t) = x(t) - \pi$, respectively. □

In view of the above analysis, we can state the following.

Corollary 16. *If $g(x) = \sin x$ and $\alpha+\beta > 0$, the subharmonic solutions found in Corollary 12 lie in the region between the two heteroclinic orbits $\Gamma^{\pm}$, if $\|f\|_1$ is small enough. More precisely, if x is such a solution for equation* (6) *and (x, y) is the solution of the corresponding Hamiltonian system, then*

$$-\pi < x(t) < \pi\,, \quad |y(t)| < Y(x(t))\,, \qquad \textit{for every } t \in \mathbb{R}\,,$$

where $Y(x)$ is the function defined in (24).

We now consider the equation

$$(|x'|^{p-2}x')' + (\alpha + \beta|x'|^p) \sin x = 0\,, \tag{25}$$

with $p > 1$, and the corresponding Hamiltonian function

$$H_2(x, y) = \begin{cases} e^{-q\beta \cos x}\dfrac{1}{q}\Big(\dfrac{\alpha}{\beta} + e^{q^2\beta \cos x}|y|^q\Big)\,, & \beta \neq 0\,, \\ \dfrac{1}{q}|y|^q + \alpha(1 - \cos x)\,, & \beta = 0\,, \end{cases} \tag{26}$$

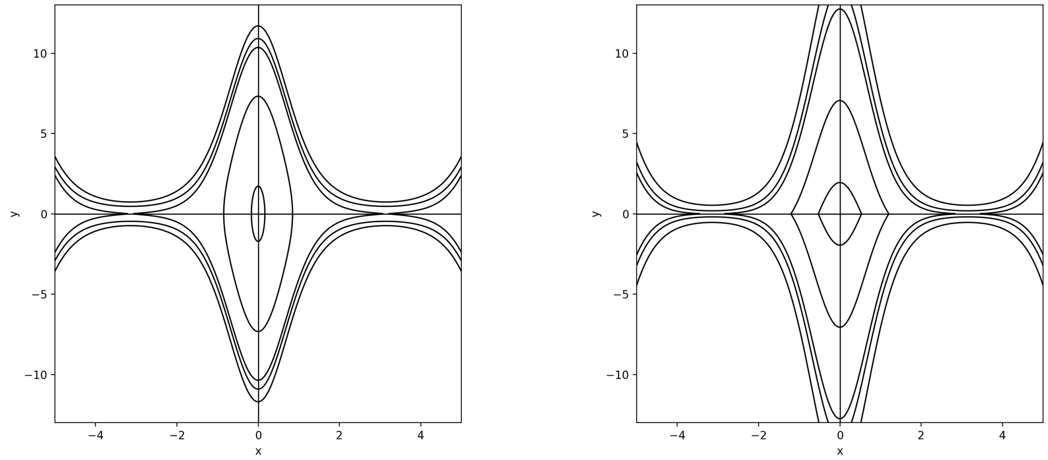

Figure 2: *The level lines of* (26) *with* $\alpha = 2$ *and* $\beta = -1$. *On the left side* $q = 2$ *(i.e.,* $p = 2$*), while on the right side* $q = 1.01$ *(i.e.,* $p = 101$*).*

where $(1/p) + (1/q) = 1$. Notice that, when $\beta \neq 0$, referring to Lemma 8, we have chosen $g(x) = q\beta \sin x$ and $G(x) = -q\beta \cos x$, while $\phi(v)$ and $\xi(v)$ are defined as in (11) so that, by (13), we have $\eta(w) = (1/q)(\alpha/\beta + |w|^q)$. The level lines of H_2 are depicted in Figure 2.

Proposition 17. *Assume* $\alpha > 0$. *The critical energy level through* $(\pi, 0)$ *contains two invariant curves* $\Gamma^{\pm}$ *connecting* $(-\pi, 0)$ *to* $(\pi, 0)$ *in opposite time directions, symmetrically, given on* $[-\pi, \pi]$ *by the graphs* $y = \pm Y(x)$*, where*

$$Y(x)^q = \begin{cases} \dfrac{\alpha}{\beta}\, e^{-q^2\beta\cos x}\Big(e^{q\beta(1+\cos x)} - 1\Big)\,, & \beta \neq 0\,, \\ q\alpha(1 + \cos x)\,, & \beta = 0\,. \end{cases} \tag{27}$$

If $1 < p \leq 2$ *(i.e.,* $q \geq 2$*), the curves* $\Gamma^{\pm}$ *are heteroclinic orbits: if* $(x(t), y(t))$ *is a solution on* Γ^+*, then*

$$\lim_{t\to-\infty}(x(t), y(t)) = (-\pi, 0)\,, \qquad \lim_{t\to+\infty}(x(t), y(t)) = (\pi, 0)\,.$$

More precisely:

- *If* $p = 2$ *(i.e.,* $q = 2$*), then*

$$x(t) + \pi \sim C\, e^{-\sqrt{\alpha}\,|t|} \quad \textit{as} \;\; t \to -\infty\,,$$

$$\pi - x(t) \sim C\, e^{-\sqrt{\alpha}\,t} \quad \textit{as} \;\; t \to +\infty\,,$$

for some $C > 0$.

- *If* $1 < p < 2$ *(i.e.,* $q > 2$*), then*

$$x(t) + \pi \sim C_\alpha\, |t|^{-\frac{p}{2-p}} \quad \textit{as} \;\; t \to -\infty\,,$$

$$\pi - x(t) \sim C_\alpha t^{-\frac{p}{2-p}} \quad as \;\; t \to +\infty\,,$$

where

$$C_\alpha = \Big(\frac{2-p}{p}\Big)^{\frac{p}{p-2}} \Big(\frac{\alpha p}{2(p-1)}\Big)^{\frac{1}{p-2}}\,.$$

On the other hand, if $p > 2$ *(i.e.,* $1 < q < 2$*), then the equilibria are reached in finite time: if* $(x(t), y(t))$ *is a solution on* Γ^+*, there exist* $t_1 < t_2$ *in* $\mathbb{R}$ *such that*

$$\lim_{t\to t_1^+} (x(t), y(t)) = (-\pi, 0)\,, \qquad \lim_{t\to t_2^-} (x(t), y(t)) = (\pi, 0)\,,$$

and

$$x(t) + \pi \sim K_\alpha (t - t_1)^{\frac{p}{p-2}} \quad as \;\; t \to t_1^+\,,$$
$$\pi - x(t) \sim K_\alpha (t_2 - t)^{\frac{p}{p-2}} \quad as \;\; t \to t_2^-\,,$$

where

$$K_\alpha = \Big(\frac{p-2}{p}\Big)^{\frac{p}{p-2}} \Big(\frac{\alpha p}{2(p-1)}\Big)^{\frac{1}{p-2}}\,.$$

A similar dynamics occurs on Γ^-*.*

Proof. It is easy to see that the equilibria are $(k\pi, 0)$. For $\beta = 0$ the critical level through $(\pi, 0)$ is

$$H_2(x, y) = H_2(\pi, 0) \quad \Leftrightarrow \quad \frac{1}{q}|y|^q + \alpha(1 - \cos x) = 2\alpha\,,$$

which gives two invariant graphs $y = \pm Y(x)$ joining $(-\pi, 0)$ and $(\pi, 0)$.

For $\beta \neq 0$ the critical level through $(\pi, 0)$ is

$$H_2(x, y) = H_2(\pi, 0) = \frac{\alpha}{\beta q} e^{q\beta}.$$

Multiplying by $q e^{q\beta \cos x}$ one obtains

$$\frac{\alpha}{\beta} + e^{q^2\beta \cos x}|y|^q = \frac{\alpha}{\beta} e^{q\beta(1+\cos x)},$$

hence the claimed formula for $Y(x)^q$ for the two smooth branches $y = \pm Y(x)$ connecting $(-\pi, 0)$ and $(\pi, 0)$.

Let us focus on $(\pi, 0)$. Along Γ^+ one has $x' = e^{(q^2-q)\beta \cos x} y^{q-1}$ and $y = Y(x)$. Let $\delta = \pi - x$. Since

$$\cos(\pi - \delta) = -1 + \frac{\delta^2}{2} + o(\delta^2)\,,$$

one gets the expansion, both for $\beta = 0$ and $\beta \neq 0$,

$$Y(x)^q \sim \frac{\alpha q}{2}\, e^{q^2\beta}\, \delta^2,$$

which implies

$$\delta' \sim -\Big(\frac{\alpha q}{2}\Big)^{\frac{1}{p}} \delta^{2/p}.$$

The conclusion follows, after analyzing the differential equation in the different cases.

The above argument can be symmetrically adapted to study the behavior at $(-\pi, 0)$. The proof of the proposition is thus completed. □

We thus get the analogue of Corollary 16 when $p = 2$.

Corollary 18. *If $g(x) = \sin x$ and $\alpha > 0$, the subharmonic solutions found in Corollary 13 lie in the region between the two heteroclinic orbits $\Gamma^{\pm}$, if $\|f\|_1$ is small enough. More precisely, if x is such a solution for equation* (12) *and (x, y) is the solution of the corresponding Hamiltonian system, then*

$$-\pi < x(t) < \pi\,, \quad |y(t)| < Y(x(t))\,, \qquad \textit{for every } t \in \mathbb{R}\,,$$

where $Y(x)$ is the function defined in (27) *with $q = 2$.*

For $1 < p < 2$ the dynamics near the origin is strongly slowed down: the angular velocity in the phase plane vanishes at the origin and the period of small oscillations diverges. It is the same situation we encounter near the heteroclinics, hence apparently there will be no twist. On the contrary, however, there is an intermediate orbit whose period $\bar{\tau}$ is minimal among all those solutions whose orbits lie between the two heteroclinics. Hence, if we take $mT > \bar{\tau}$, we find two twist situations: one inside the orbit having period $\bar{\tau}$ and one outside, and Theorem 1 applies. We have thus proved the following.

Theorem 19. *Assume $1 < p < 2$ and let α and β be real constants, with $\alpha > 0$. There exists a positive integer $\overline{m}$ with the following property: for every $m \geq \overline{m}$ there is an $\varepsilon_m > 0$ such that, if $\|f\|_1 \leq \varepsilon_m$ and* (1) *holds, equation*

$$(|x'|^{p-2}x')' + (\alpha + \beta|x'|^p)\sin x = f(t) \tag{28}$$

has at least four periodic solutions $x_1(t), \ldots, x_4(t)$ having minimal period mT. Moreover, for $j = 1, \ldots, 4$, the curves $t \mapsto (x_j(t), x_j'(t))$ do not cross the origin, and rotate around it exactly once in the time interval $[0, mT]$.

Concerning the case $p > 2$, let us denote by $\tau_{\alpha,\beta}$ the time needed to connect $(-\pi, 0)$ to $(\pi, 0)$. Since the period of the solutions near the origin approaches 0 and the period of those near the separatrix is almost $2\tau_{\alpha,\beta}$, we can easily prove the following result.

Theorem 20. *Assume $p > 2$ and let α and β be real constants, with $\alpha > 0$. If m is a positive integer such that*

$$mT < 2\tau_{\alpha,\beta}\,,$$

then the same conclusion of Theorem 11 holds for equation (28).

5 Periodic solutions of the second kind

In this section, assuming $f : \mathbb{R} \to \mathbb{R}$ to be locally integrable and T-periodic, satisfying (1), we study the existence of the so-called *periodic solutions of the second kind*, i.e., those satisfying

$$x(t+T) = x(t) + 2\pi k\,, \qquad \text{for every } t \in \mathbb{R}\,, \tag{29}$$

for some integer k. These are sometimes referred to as *running solutions* (see, e.g., [11, 18]). Clearly, if $k = 0$, we recover the usual periodicity. Notice that (29) is equivalent to ask that the function $\hat{x}_k(t) = x(t) - (2\pi k/T)t$ be T-periodic. We can then apply a variant of Theorem 1, where the twist assumption (5) is replaced by

$$\begin{cases} z(0) \in \mathcal{G}_a \quad \Rightarrow \quad x(T) - x(0) < 2\pi k\,, \\ z(0) \in \mathcal{G}_b \quad \Rightarrow \quad x(T) - x(0) > 2\pi k\,, \end{cases} \tag{30}$$

so to obtain this type of solutions.

Here is our result.

Theorem 21. *Under the assumptions of either Theorem 2 or Theorem 4, for every integer k in the interval $\left]-\frac{T}{2\pi}\rho, \frac{T}{2\pi}\rho\right[$ equation* (3) *has at least two geometrically distinct periodic solutions of the second kind satisfying* (29)*.*

The proof is a straightforward modification of the proof of Theorems 2 and 4. For the relativistic equation (6), we have the following.

Corollary 22. *Assume $g : \mathbb{R} \to \mathbb{R}$ to be 2π-periodic, with $\int_0^{2\pi} g(x)\,dx = 0$. Then, for any constants α and β and every integer k in the interval $\left]-\frac{T}{2\pi}, \frac{T}{2\pi}\right[$, equation* (6) *has at least two geometrically distinct periodic solutions of the second kind satisfying* (29)*, provided that f verifies* (1)*.*

Concerning the p-Laplacian equation (7), the corresponding corollary is the following.

Corollary 23. *Let $g : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, with $\int_0^{2\pi} g(x)\,dx = 0$. Then, for any constants α and β and every $k \in \mathbb{Z}$, equation* (7)*, with $p > 1$, has at least two geometrically distinct periodic solutions of the second kind satisfying* (29)*, provided that f verifies* (1)*.*

And here is the corresponding statement for equation (12).

Corollary 24. *For any constants α, β and every $k \in \mathbb{Z}$, equation* (12) *has at least two geometrically distinct periodic solutions of the second kind satisfying* (29)*, provided that f verifies* (1)*.*

When $g(x) = \sin x$, if we consider a small forcing term, more information about the periodic solutions of the second kind can be provided. The heteroclinic orbits in equation (22) permits us to say that for every integer k in the interval $\left]0, \frac{T}{2\pi}\rho\right[$, if $\|f\|_1$ is small enough, the equation

$$\left(\frac{x'}{\sqrt{1-|x'|^2}}\right)' + \left(\alpha + \frac{\beta}{\sqrt{1-|x'|^2}}\right)\sin x = f(t)$$

has at least two geometrically distinct periodic solutions of the second kind, satisfying (29), with $x'(t) > 0$ for every $t \in \mathbb{R}$. Indeed, it is possible to find two level lines ℓ_* and ℓ_{**} of (23), lying above the upper heteroclinic, such that, parametrizing them as $x \mapsto (x, a(x))$ and $x \mapsto (x, b(x))$, respectively, with $a(x) < b(x)$, so that $\ell_* = \mathcal{G}_a$ and $\ell_{**} = \mathcal{G}_b$, if $(x_*(t), y_*(t))$ is a solution of (22) lying on ℓ_*, then

$$x_*(t+T) - x_*(t) < 2\pi k\,, \qquad \text{for every } t \in \mathbb{R}\,, \tag{31}$$

while if $(x_{**}(t), y_{**}(t))$ is a solution of (22) lying on ℓ_{**}, then

$$x_{**}(t+T) - x_{**}(t) > 2\pi k\,, \qquad \text{for every } t \in \mathbb{R}\,.$$

The twist property (30) then persists under small perturbations, hence the corresponding variant of Theorem 1 applies, providing us the two solutions whose orbits lie above the upper heteroclinic.

Symmetrically, if the integer k in chosen the interval $\left] -\frac{T}{2\pi}\rho, 0\right[$, we will find our two solutions below the lower heteroclinic, thus having the property $x'(t) < 0$ for every $t \in \mathbb{R}$.

For equation (28) with $p = 2$ the same will be true for every positive or negative integer k. However, when $p > 2$, the heteroclinic orbits of (25) are replaced by separatrices. Then, recalling that we denote by $\tau_{\alpha,\beta}$ the time needed for a solution to travel from $-\pi$ to π, in order to have the solution $x_*(t)$ satisfying the above property (31) we will need to assume that $T < k\tau_{\alpha,\beta}$.

6 Final remarks and open problems

We conclude this paper with some suggestions for further developments.

1. **More general nonlinearities.** It would be interesting to know if our results could be extended to the case when the function $g(x)$ is replaced by a more general nonlinearity $g(t, x)$. This was done for the classical pendulum equation [22] and for the relativistic pendulum equation [5]. However, our method (as well as the one in [2]) does not seem to be applicable to this more general context.

2. **The problem with friction.** There is a large literature on the pendulum equation with a friction term (see, e.g., [19] and the references therein). Recently Amster and Cid [1] provided the existence of periodic solutions for the equation

$$\left(\frac{x'}{\sqrt{1-|x'|^2}}\right)' + cx' + \frac{\beta}{\sqrt{1-|x'|^2}}\sin x = f(t)\,.$$

It seems reasonable that this study could be extended to equations with a more general differential operator.

3. **Stability.** In [27, 28] Ortega analyzed the stability problem for the classical pendulum equation. Later on, in [30] a similar result was provided for the relativistic pendulum. It would be interesting to carry on this type of investigation also for the general equation (3).

4. **The mean curvature equation.** The case when $\phi(v) = v/\sqrt{1+|v|^2}$ in equation (3) seems more delicate. Even when ξ is constant, some extra assumptions need to be added (see, e.g., [11, 26]). We prefer not entering into this problem for the sake of briefness.

5. **Higher dimensional problems.** Extensions of the results in [22] and [5] to higher dimensional systems have been provided in [23] and [21], respectively. We now propose a possible higher dimensional version of equation (3).

If $G : \mathbb{R}^N \to \mathbb{R}$ is 2π-periodic in each variable $x_1, \dots, x_N$ and $F : \mathbb{R} \to \mathbb{R}^N$ is T-periodic, our results can be extended to systems with Hamiltonian function $H : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ defined as

$$H(t,x,y) = e^{G(x)}\eta(e^{-G(x)}y + F(t))\,.$$

Assuming (13), i.e., $\eta(w) = \xi(\phi^{-1}(w))$, computation leads to the differential equation

$$(\phi(x'))' + (x'\cdot\nabla G(x))(\phi(x') - F(t)) + \big(\xi(x') - x'\cdot(\phi(x') - F(t))\big)\nabla G(x) = f(t)\,,$$

where $f(t) = F'(t)$. A higher dimensional version of the Poincaré–Birkhoff proposed in [13] then provides the existence of at least $N+1$ geometrically distinct T-periodic solutions.

Notice that, when $F = 0$, the equilibria satisfy $\xi(0)\nabla G(x) = 0$. Hence, if $\xi(0) = 0$, all the constants $x \in \mathbb{R}^N$ are equilibria, while if $\xi(0) \neq 0$ the equilibria are the critical points of G. By its periodicity, one can think of G as being defined on the torus $\mathbb{T}^N$, and the Lusternik–Schnirelmann theory guarantees the existence of at least $\mathrm{cat}(\mathbb{T}^N) = N+1$ critical points.

When $N = 3$, by the use of the formula

$$a \times (b \times c) = (a\cdot c)b - (a\cdot b)c\,,$$

we get the equation

$$(\phi(x'))' + x' \times \big((\phi(x') - F(t)) \times \nabla G(x)\big) + \xi(x')\nabla G(x) = f(t)\,.$$

In this case, our theorem provides the existence of at least four geometrically distinct T-periodic solutions.

As a particular case, denoting by $\mathcal{B}_1$ the unitary open ball in $\mathbb{R}^3$ centered at the origin, let $\phi : \mathcal{B}_1 \to \mathbb{R}^3$ and $\xi : \mathcal{B}_1 \to \mathbb{R}$ be defined as

$$\phi(v) = \frac{v}{\sqrt{1-|v|^2}}\,, \quad \xi(v) = \frac{\alpha}{\beta} + \frac{1}{\sqrt{1-|v|^2}}\,.$$

Then, in the unforced case $F = 0$, we are led to the equation

$$\left(\frac{x'}{\sqrt{1-|x'|^2}}\right)' + x' \times \left(\frac{x'}{\sqrt{1-|x'|^2}} \times \nabla G(x)\right) + \left(\alpha + \frac{\beta}{\sqrt{1-|x'|^2}}\right)\nabla G(x) = 0\,.$$

Overall, the equation describes the motion of a relativistic particle in a non-isotropic external field, where the gradient of G generates both a longitudinal (conservative) and a transverse (gyroscopic) component of the force. This structure suggests an analogy with the Lorentz equations for a charged particle, with effective fields depending on the velocity. The presence of the parameters α and β allows one to interpolate between a more 'classical' regime (when $\beta = 0$) and a regime with a more pronounced 'relativistic' response (when $\beta \neq 0$). We do not enter into further details, for briefness.

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